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TECHNICAL NOTE

No. 932

THE NUMERICAL SOLUTION OF COMPRESSIBLE FLUID FLOW PROBLEMS

By Howard W. Emmons
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ERRATA

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Page 12, line 16: Change $\dfrac{P_{stagnation}}{p_o} = e^{+\frac{s-s_o}{R}}$

to read $\dfrac{P_{stagnation}}{p_o} = e^{-\frac{s-s_o}{R}}$

Page 16, paragraph 4, line 9: Change ". . . upper left of each net point . . ." to read ". . . upper right of each net point . . ."

Page 20, equation 25, first part of second line: Change

$-\psi_\xi'(\ln \rho/\rho_o)$ to read $-\psi_\xi'(\ln \rho/\rho_o)_\xi$

Page 25, line 14: The sentence "The balanced case is important since it insures zero rotation for the adiabatic flow of a gas from a large region of zero velocity even though the temperature is not uniform there," is incorrect. It can be shown that

$$2\omega = \frac{pq^2}{2}\,\frac{\partial \ln T_o}{\partial \psi}$$

for flow from a reservoir of nonuniform temperature T_o.

Page 26, equation (39c): Change $c_v \dfrac{dT}{T} - R\dfrac{dp}{p}$ to read $c_v \dfrac{dT}{T} - R\dfrac{d\rho}{\rho}$.

Page 28, line 7: Change q_{cr} critical velocity $= \dfrac{2}{\gamma+1}a^2 + \dfrac{\gamma-1}{\gamma+1}q^2$ to read q_{cr} critical velocity $= \left\{\dfrac{2}{\gamma+1}a^2 + \dfrac{\gamma-1}{\gamma+1}q_b^2\right\}^{1/2}$

Page 29, equation (49): Change ΔXY . . . . for any function $\varphi\,(X,Y)$, to read Δ_{xy} . . . . for any function $\varphi\,x,y$.

Page 29, sentence following equation (54): Change "equation (48) follows." to read "equation (49) follows."

Page 30, equation (22): Change the term $\psi_\eta\left(\ln\frac{q}{q_i}\right)$ to read $\psi_\eta\left(\ln\frac{q}{q_i}\right)_\eta$

Figure 20, equation for figure reads:

$$\frac{\rho q}{\rho_o a_o} = \left(\frac{p}{p_o}\right)^{\frac{1}{\gamma}} e^{\frac{s-s_o}{\gamma R}} \left\{\frac{2}{\gamma-1}\left(1-\left(\frac{p}{p_o}\right)^{\frac{\gamma-1}{\gamma}} e^{-\frac{\gamma-1}{\gamma}\frac{s-s_o}{R}}\right)\right\}^{1/2}$$

Change to read:

$$\frac{\rho q}{\rho_o a_o} = \left(\frac{p}{p_o}\right)^{\frac{1}{\gamma}} e^{-\frac{\gamma-1}{\gamma}\frac{s-s_o}{R}} \left\{\frac{2}{\gamma-1}\left(1-\left(\frac{p}{p_o}\right)^{\frac{\gamma-1}{\gamma}} e^{+\frac{\gamma-1}{\gamma}\frac{s-s_o}{R}}\right)\right\}^{1/2}$$

Figure 22, equation for figure reads:

$$\frac{\rho q}{\rho_o a_o} = \left\{\frac{2}{\gamma-1}\left[1-\left(\frac{p\,\rho}{p_o\rho_o}\right)^{\frac{\gamma-1}{\gamma+1}} e^{2\frac{\gamma-1}{\gamma+1}\frac{s-s_o}{R}}\right]\right\}^{1/2} \frac{p\,\rho}{p_o\rho_o}\, e^{-\frac{\gamma-1}{\gamma+1}\frac{s-s_o}{R}}$$

Change to read:

$$\frac{\rho q}{\rho_o a_o} = \left\{\frac{2}{\gamma-1}\left[1-\left(\frac{p\,\rho}{p_o\rho_o}\right)^{\frac{\gamma-1}{\gamma+1}} e^{2\frac{\gamma-1}{\gamma+1}\frac{s-s_o}{R}}\right]\right\}^{1/2} \frac{p\,\rho}{p_o\rho_o}^{\frac{1}{\gamma+1}} e^{-\frac{\gamma-1}{\gamma+1}\frac{s-s_o}{R}}$$

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SUMMARY

Numerical methods have been developed for obtaining the steady, adiabatic flow field of a frictionless, perfect gas about arbitrary two-dimensional bodies. The solutions include the subsonic velocity regions, the supersonic velocity regions, and the transition compression shocks, if required. Furthermore, the rotational motion and entropy changes following shocks are taken into account. Extensive use is made of the relaxation method.

In this report the details of the methods of solution are emphasized so as to permit others to solve similar problems. Solutions already obtained are mentioned only by way of illustrating the possibilities of the methods described.

The methods can be applied directly to wind tunnel and free air tests of arbitrary airfoil shapes at subsonic, sonic, and supersonic speeds.

INTRODUCTION

The knowledge of the flow of incompressible fluids about bodies, especially airfoil shapes, has been greatly advanced by the interpretation of good experimental results in the light of theoretical predictions. The first successful, easiest, and most widely useful theoretical results have come from a consideration of the two-dimensional irrotational flow of an incompressible perfect fluid.

The knowledge of the flow of compressible fluids has made good progress in exactly the same way for two widely separated conditions. First, linearization and perturbation methods yield helpful information up to moderate velocities, a Mach number from 0.5 to 0.7 depending upon the body. Second, the method of characteristics throws a great deal of light on completely supersonic flows. Analytical difficulties have to date prevented the extension of theoretical results to many flow problems in which both subsonic and supersonic velocity regions occur. A. Chaplygin, Ringleb, and Tollmien (references 1, 2, and 3) have obtained a few suggestive exact solutions involving subsonic and supersonic velocity regions. Meyer, Taylor, and Görtler (references 4, 5, and 6) have studied in a crude approximate way the passage through sonic velocity in a nozzle. None of these methods of solution is able to fit arbitrary body shapes and is completely incapable of predicting the occurrence, location, and shape of compression shocks.

When shocks are present in a solution, the asumption of irrotationality of the flow of a compressible fluid is, in general, no longer correct. Special forms of the equation describing the rotational motion of a gas have been discussed by Friedrichs and Crocco (references 7 and 8). A consideration of the complexity of these equations together with the almost insurmountable analytical difficulties encountered in attempting solutions of adiabatic, frictionless, irrotational, shock-free flow makes it obvious that analytical solutions of general high velocity problems are not likely to be found in the near future.

A new, rather general idea was introduced into the numerical solution of difficult problems during the nineteen thirties. R. V. Southwell's relaxation method (references 9, 10, 11) permits the solution of problems of the flow of incompressible, perfect fluids with great ease, and is readily adapted to the solution of subsonic problems of adabatic, frictionless (not necessarily irrotational) flow. The relaxation method is not directly applicable to supersonic velocity regions, but an alternative procedure based upon the use of the finite difference equations has been worked out. Finally, the fitting together of the subsonic and supersonic regions, adjusting their shape and size with compression shocks, if necessary, is accomplished by a combination of methods.

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SYMBOLS

a acoustic velocity

c_p; c_v specific heat at constant pressure and volume, respectively

C constant

D reference dimension setting physical scale of airfoil or tunnel

h specific enthalpy

L distance along streamline

$M = \frac{q}{a}$ Mach number

n normal distance

p pressure

q velocity (components u, v)

Q residual to be liquidated

R gas constant

r radius of curvature of streamline

s specific entropy

T absolute temperature

u velocity component in x direction

U velocity of undisturbed stream

v velocity component in y direction

x,y coordinates in physical plane

$\gamma = \frac{c_p}{c_v}$ isentropic exponent

δ lattice spacing in computations

Δ change of a quantity or the Laplace operator

φ Scalar variable

ψ stream function

ψ_0 constant

ψ' dimensionless stream function

η stream function for incompressible fluid

ξ velocity potential for incompressible fluid

ρ mass density

ω rate of rotation

Subscripts

i incompressible fluid

$x, y, \xi, \eta,$ denote differentiation in the corresponding direction

$X = \frac{x}{D}$, $Y = \frac{y}{D}$ denote differentiation with respect to dimensionless coordinate in the physical plane

1,2,3,4,0 Lattice points

o isentropic stagnation conditions for undisturbed stream

RELAXATION SOLUTION OF THE FLOW OF INCOMPRESSIBLE FLUIDS

The two-dimensional irrotational flow of an incompressible fluid is described by either of the equations

$$\Delta\eta = 0 \quad (\eta \text{ is the stream function}) \qquad \text{a)}$$
$$\Delta\xi = 0 \quad (\xi \text{ is the velocity potential}) \qquad \text{b)} \qquad (1)$$

where the velocity components are given by

$$u_i = \frac{\partial \xi}{\partial\left(\frac{x}{D}\right)} = \xi_{\left(\frac{x}{D}\right)} = \eta_{\left(\frac{y}{D}\right)} \qquad \text{a)}$$

$$v_i = \xi_{\left(\frac{y}{D}\right)} = -\eta_{\left(\frac{x}{D}\right)} \qquad \text{b)} \qquad (2)$$

$$q_i = (u_i^2 + v_i^2)^{1/2} \qquad \text{c)}$$

To find the flow about a given airfoil, it is necessary to find a solution to one of the equations (1) subject to the boundary conditions that the surface of the airfoil is a streamline and conditions at infinity are uniform. Thus in figure 1, the boundary conditions would be:

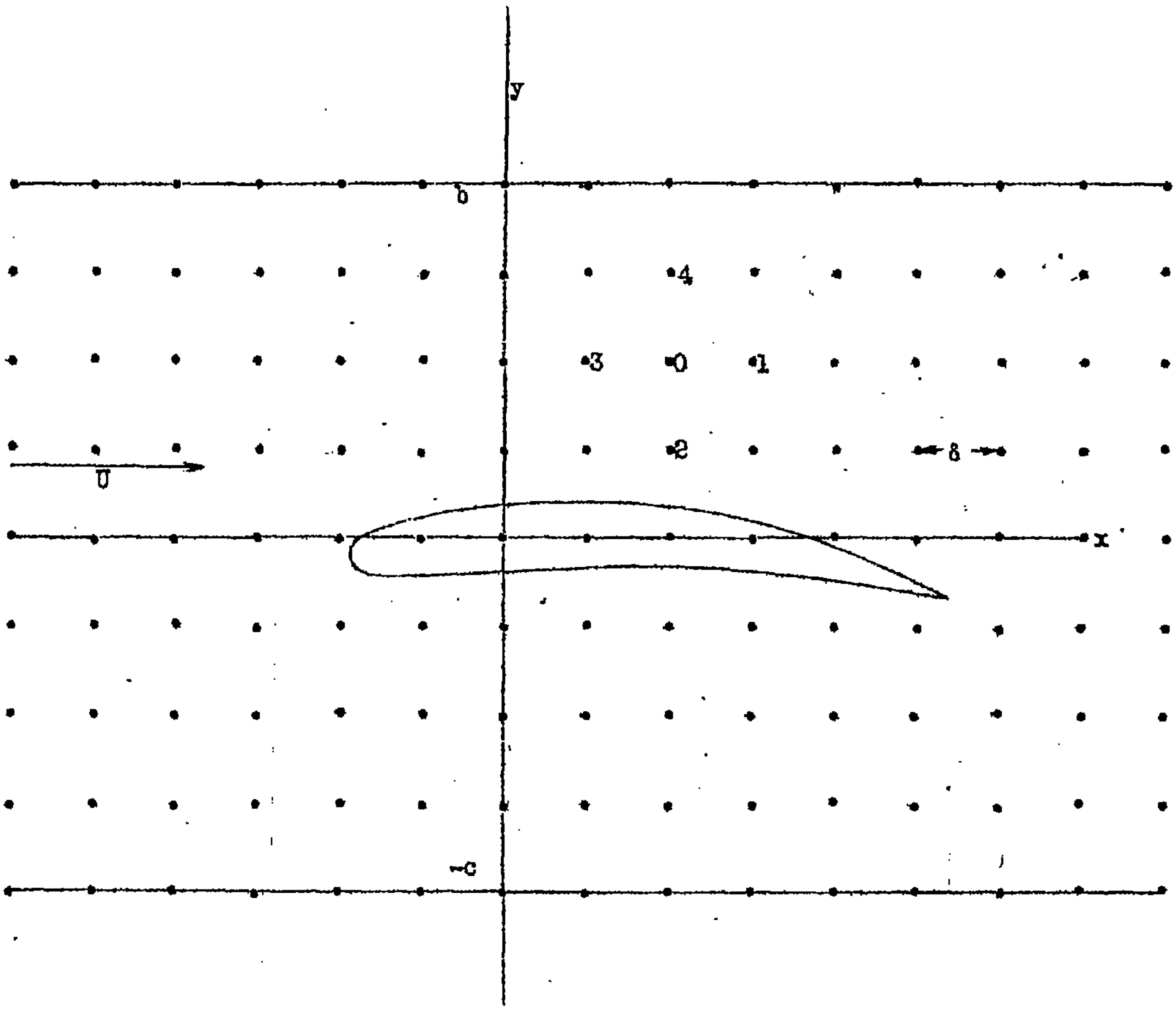

Figure 1

For the stream function

$$\left.\begin{aligned} \eta &= \text{a constant on the airfoil} \\ \eta &= -\eta_c \text{ at } y = -c \\ \eta &= \eta_b \text{ at } y = b \\ \eta &= \frac{Uy}{D} \text{ for } x \longrightarrow \pm\infty \end{aligned}\right\} \quad (3a)$$

For the velocity potential

$$\left.\begin{aligned} \frac{\partial \xi}{\partial\left(\frac{n}{D}\right)} &= 0 \text{ along airfoil} \\ \frac{\partial \xi}{\partial\left(\frac{n}{D}\right)} &= 0 \text{ at } y = -c,\ b \\ \xi &\longrightarrow \frac{Ux}{D} \text{ for } x \longrightarrow \pm\infty \end{aligned}\right\} \quad (3b)$$

The Joukowski condition of no flow around the trailing edge must be added.

To solve equations (1) subject to conditions (3) for an arbitrary airfoil shape, the relaxation method is by far the most practical. Christopherson and Southwell in reference 10 discuss the method in a general way; Emmons in reference 11 gives a more detailed application to the solution of the Laplace equation. The method is outlined below.

The desired equation, say (1a), is written in the approximate finite difference form (see reference 10 or 11):

$$\eta_1 + \eta_2 + \eta_3 + \eta_4 - 4\eta_0 = 0 \quad (4)$$

where η_0 is the value of η at an arbitrary point in a square net of points (see fig. 1) and η_1, η_2, η_3, η_4 are the values of η at the four surrounding points.

If by some process, values of η were attached to each point, equation (4) would immediately show whether or not they approximated a solution of Laplace's equation. If the attached values do not satisfy (4), they define a residual Q at each point.

$$\eta_1 + \eta_2 + \eta_3 + \eta_4 - 4\eta_0 = Q_0 \qquad (5)$$

Observe that a change of value of η_0 by 1, all other η's held constant, would change the value of the Q's at various points as shown in figure 2. Thus the residuals may be moved at will from any given point to the surrounding points. This process is physically equivalent to removing restraints from a tension net; hence the term "relaxation method." Figure 2 is called by Southwell the relaxation pattern. It gives at a glance the influence coefficients for the effect of changes of η on the residuals. This relaxation process is followed step by step until all interior Q are zero and the boundary values are as desired.

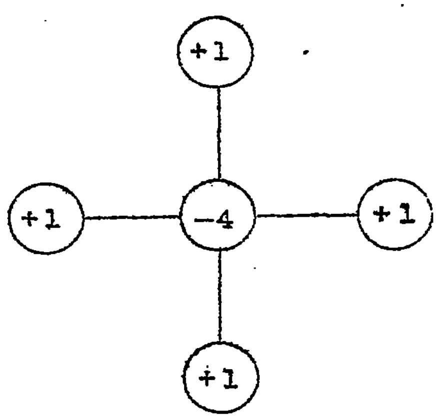

Figure 2

To calculate the flow about an airfoil, take the following steps:

1. Draw the airfoil and flow region to a scale such that the distance between net points, as δ in figure 1, is about $1\frac{1}{2}$ inches. Do not use too many points at the start. Those shown in figure 1 are adequate. For greater accuracy more points can be added in important regions, as near the airfoil, during the course of solution.

2. With the boundary conditions in mind, guess values of η at the net points, and compute the residuals. To aid the accuracy of guessing, a freehand sketch of streamlines and potential lines is sometimes useful. Use whole values of η ranging from say 0 to 1000. It is convenient to record at each net point values as indicated in figure 3.

Q_o ●	η guess
Q_1	$\Delta\eta_1$
Q_2	$\Delta\eta_2$
etc.	etc.
Q_{final}	η_{final}

Figure 3

3. The residuals are relaxed, each time recording at each point the change in η and the resultant Q. In this way the points at which the residual is largest can be spotted at a glance and relaxed next. Change η by simple whole numbers only.

4. After all Q have values between ± 2 (move decimal to position of desired accuracy) add changes of η to get the final value (fig. 3) at each point.

5. Recompute Q by equation (5) to locate any computation errors. Relax resultant Q if any.

6. If the solution is not accurate enough, additional points are added where needed. In figure 1 many more points are needed near the airfoil. The process 1 through 5 is repeated as previously mentioned.

7. The required results - for example, pressure distribution - can be computed by use of equations (2) and Bernoulli's equation. The accuracy of all the results can be at any time improved by adding more points to the net used in the solutions.

The boundary values as given will be information about values of the desired function or values of the normal derivative of the desired function as in equation (3). When the physical boundary runs between net points, it is sufficiently accurate to set values at the nearest net points by linear interpolation or extrapolation.

As will be described in a following section, the flow of a compressible fluid is best accomplished by making use of the streamlines and potential lines of the irrotational flow of an incompressible fluid about the same body.

Differential Equations and Boundary Conditions

for the Adiabatic Flow of a Frictionless Perfect Gas

The motion of a compressible fluid is described by three laws of nature: namely, conservation of matter, energy, and momentum together with the properties of the fluid and the boundary conditions of the particular problem on hand. The second law of thermodynamics makes a restriction on the type of discontinuity (shock wave) that can occur.

In the following, the fluid will be taken as a frictionless, perfect gas. The flow will be assumed adiabatic. Thus in the absence of compression shocks the flow will be isentropic. The changes of entropy in the compression shocks will be considered in detail later.

If, in addition to the assumption of an adiabatic flow, steady flow is assumed, the energy equation states that

$$h_o = h + \frac{q^2}{2} \tag{6}$$

is constant along a streamline but may differ in an arbitrary way from one streamline to the next. For all cases assuming uniform conditions at infinity, the stagnation enthalpy, h_o, is constant everywhere. This assumption is usually adequate but, if not, it would not materially complicate the method of solution.

The continuity equation in rectangular, x, y coordinates is

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0 \tag{7}$$

This equation permits the introduction of the stream function ψ defined by

$$\rho u = \frac{\partial \psi}{\partial y} \equiv \psi_y, \quad \rho v = - \frac{\partial \psi}{\partial x} \equiv - \psi_x \tag{8}$$

The substance of the equations of motion for an adiabatic, frictionless flow are summed up in the equations (see appendix 1 for derivation)

$$\omega \equiv \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) = \frac{1}{2} \left\{ \frac{p}{R} \frac{\partial s}{\partial \psi} - \rho \frac{\partial h_o}{\partial \psi} \right\} \tag{9}$$

where the entropy s and the stagnation enthalpy h_o are constant along a streamline. Generally, s and h_o are both constant everywhere from which the flow is seen to be irrotational.

Equations (8) substituted into equation (9) yields the fundamental differential equation to be solved.

$$\frac{\partial}{\partial x}\left(\frac{1}{\rho} \frac{\partial \psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{\rho} \frac{\partial \psi}{\partial y}\right) = -2\omega \tag{10a}$$

The following form of equation (10) is generally more convenient for numerical solution,

$$\psi_{xx} + \psi_{yy} - \psi_x (\ln \rho)_x - \psi_y (\ln \rho)_y + 2\omega\rho = 0 \tag{10b}$$

If the density is constant and the rotation is zero, this equation reduces to Laplace's equation as used in the previous section. The values of the density ratio, ρ/ρ_o, to be used in equation (10) are obtained from (see appendix 2 for derivation)

$$\rho/\rho_o = \left\{1 - \frac{\gamma-1}{2}\left(\frac{q}{a_o}\right)^2\right\}^{\frac{1}{\gamma-1}} e^{-\frac{s-s_o}{R}} \qquad (11)$$

and

$$\rho q = \left\{(\rho u)^2 + (\rho v)^2\right\}^{1/2} = \left\{\psi_x^{\,2} + \psi_y^{\,2}\right\}^{1/2} \qquad (12)$$

Thus the density is given during the course of solution by a relation

$$\ln\ \rho/\rho_o = f\left(\frac{q\rho}{a_o\rho_o}, \frac{s-s_o}{R}\right) \qquad (13)$$

This relation is plotted as computation figure 12.

It should be noted that the entropy increase is simply related to the change in total pressure in the absence of heat transfer and friction. The relation $\frac{p_{stagnation}}{p_o} = e^{-\frac{s-s_o}{R}}$ has been plotted as computation figure 25. This relation is only correct when the stagnation enthalpy is constant everywhere. The $\frac{s-s_o}{R}$ term is used directly in this report in spite of the experimental significance of the stagnation pressure ratio because of its ease of use and its more fundamental nature in equation (9).

During the course of a solution the values of ψ_x and ψ_y are periodically introduced into equation (12); $\ln\ \rho/\rho_o$ is then evaluated from computation figure 12 and is used to correct the density terms in equation (10).

The boundary conditions, as for incompressible fluids, are commonly taken as uniform properties and velocity at infinity and a certain few streamlines specified by the surfaces of bodies (airfoil) and flow passages (wind tunnel).

It was noted in the previous section that the solution of the flow of an incompressible fluid about an airfoil by the relaxation method required special attention to the boundary conditions when the net points did not fall directly on the boundary itself. The flow of a compressible fluid, especially near the speed of sound, involves so many difficulties that it is desirable to avoid the boundary condition trouble. This is easily done by using as a coordinate system the streamlines (η = const) and potential lines (ξ = const) for the irrotational flow of a perfect incompressible fluid about the same airfoil. Figure 4 shows the airfoil in the real plane and the simple straight line boundaries required in the transformed plane. Another advantage of these coordinates can be anticipated since the compressible fluid streamlines will not deviate too greatly from the incompressible streamlines (η const).

This transformation of coordinates is conformal and for any conformal transformation equations (10 a,b) become (see appendix 3).

$$\frac{\partial}{\partial \xi}\left(\frac{1}{\rho}\frac{\partial \psi}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(\frac{1}{\rho}\frac{\partial \psi}{\partial \eta}\right) = -\frac{2D^2\omega}{q_1^2} \tag{14a}$$

or

$$\psi_{\xi\xi} + \psi_{\eta\eta} - \psi_\xi (\ln \rho)_\xi - \psi_\eta (\ln \rho)_\eta + \frac{2D^2\omega\rho}{q_1^2} = 0 \tag{14b}$$

The differential equation for ψ has the same formal appearance in the physical plane and in the transformed plane (since for nearly all work the ω term is negligible). An important difference appears in the determination of the compressible fluid density during the course of the solution. In place of equation (12), the following equation is used (see appendix 3):

$$\rho q = \frac{q_1}{D}\left(\psi_\xi^2 + \psi_\eta^2\right)^{1/2} \tag{15}$$

The density is again calculated by equation (13), computation figure 12.

To determine, by equation (15), the value of the fluid density at any point requires at that point a knowledge of the ψ gradient. For a solution of the problem by a net of points there is some error involved in evaluating the gradient. The simplest reasonably accurate procedure is to calculate, for example, ψ_η at a given point as the difference between the values of ψ at the preceding and following points divided by the corresponding change of $\eta = 2\delta$. This method works well at all points away from the boundary. For boundary points there is no "next" point from which to get a slope. Of course, any method of determining an approximate value of $l n \rho$ would be satisfactory so long as the value would approach the correct value as the net interval $\delta \longrightarrow 0$. However, it is very desirable to choose a method of finding $l n \rho$ at the boundary as accurately as possible so that a relatively coarse net (large δ) will give as accurate a result as possible. The following procedure gives very good results.

Observe, first, that at the boundary compressible (ψ) and incompressible (η) streamlines coincide and hence the radius of curvature of these streamlines is the same. The kinematic relation for the rotation of fluid elements (equation (29), appendix 1) gives

$$\frac{1}{r} = -\frac{\partial l n q}{\partial n} - \frac{2\omega}{q} = -\frac{\partial l n q_i}{\partial n} \qquad (16)$$

Since ω is generally negligible, this equation shows that along the normal to the boundary; that is, along constant ξ lines, the ratio of compressible to incompressible fluid velocity (q/q_i) is constant. This relation is very good from boundary points to those next along the ξ constant lines. In this way accurate values of q at the boundary are computed and thus $l n \rho$ by computation figure 13.

Relaxation Solution of Subsonic Flow Problems

In the relaxation solution of a non-linear equation such as equation (14b), there are several possible procedures, the relative excellence of which depends upon the relative magnitude of the various terms. The following method has been found very satisfactory.

The equation (14b) for the stream function ψ is put into a dimensionless form which permits ready change of scales. Let

$$\psi = \rho_o \, a_o \, D \, \psi_o \, \psi' \qquad (17)$$

where ψ_o is a dimensionless constant to be chosen by the computer. By equations (14 a,b)

$$\frac{\partial}{\partial \xi}\left(\frac{\rho_o}{\rho} \frac{\partial \psi'}{\partial \xi}\right) + \frac{\partial}{\partial \eta}\left(\frac{\rho_o}{\rho} \frac{\partial \psi'}{\partial \eta}\right) = - \frac{2 \, D \omega}{a_o \, q_i^2 \, \psi_o} \qquad \text{a)}$$

or (18)

$$\psi'_{\xi\xi} + \psi'_{\eta\eta} - \psi'_{\xi} (ln \, \rho/\rho_o)_\xi - \psi'_{\eta} (ln \, \rho/\rho_o)_\eta + \frac{2 D \omega \rho}{a_o \, q_i^2 \, \rho_o \, \psi_o} = 0 \qquad \text{b)}$$

Equation (15) is also altered to

$$\frac{\rho q}{\rho_o a_o} = q_i \psi_o \, (\psi_\xi'^2 + \psi_\eta'^2)^{1/2} \qquad (19)$$

The equation (18b) for the stream function ψ' is put into finite difference form as follows:

$$\psi'_1 + \psi'_2 + \psi'_3 + \psi'_4 - 4\psi'_o - \frac{(\psi'_1 - \psi'_3)(ln \, \rho/\rho_{o1} - ln \, \rho/\rho_{o3})}{4}$$

$$- \frac{(\psi'_4 - \psi'_2)(ln \, \rho/\rho_{o4} - ln \, \rho/\rho_{o2})}{4} + \frac{D \omega \rho \delta}{a_o q_i^2 \, \rho_o} = Q_o \qquad (20)$$

and equation (19) becomes

$$\frac{\rho q}{\rho_0 a_0} = q_i\left\{\left(\psi'_1 - \psi'_3\right)^2 + \left(\psi'_4 - \psi'_2\right)^2\right\}^{1/2} \qquad (21)$$

where ψ_0 has been chosen equal to 2δ for convenience.

The best procedure found to date for the solution of equation (20) when the velocities are subsonic is to treat the last three terms as corrections. The solution proceeds from assumed values of ψ' to the removal of the Q by the relaxation pattern based upon the first five terms; that is, the same relaxation pattern as for incompressible fluids (fig. 2). The Q are, however, computed correctly by equation (20). Periodically during the solution the Q are recomputed by equation (20) to take account of the changing values of $\log \rho/\rho_0$ and ω terms.

In the following list, the various steps are illustrated by the solution of the flow of a compressible fluid through a hyperbolic channel (fig. 5).

1. Calculate the flow of an incompressible fluid through the same flow system using, if necessary, the relaxation method as described in the section, Relaxation Solution of Incompressible Fluid Flows. Of course, if an analytical solution is known, this can be used. In figure 5 is shown the streamlines and velocity potential lines of the flow of an incompressible fluid through a hyperbolic channel.

2. Draw the flow region on ξ, η coordinates to a sufficiently large scale to make room for the following steps of computation to be recorded at each net point. Figure 6 shows (1/4 of) the hyperbolic channel drawn in the ξ, η plane. Of course, any shape channel would fall in the same region of the ξ, η plane. The important numbers obtained in step 1 are the dimensionless incompressible fluid velocities, q_i. These are plotted in figure 5 and are recorded at the upper right ~~left~~ of each net point (fig. 6). The q_i are dimensionless since the velocity at the center of the passage $\xi = 0$, $\eta = 0$ was taken as 1. If an airfoil with lift is to be treated, the ξ, η plane would appear as in figure 4 with an upper and lower half discontinuous across the half-line $\eta = 0$, $\xi > 0$.

3. Choose the desired boundary conditions for the problem. This is not as easy a matter as for incompressible fluids where necessary and sufficient conditions for the existence of a solution of Laplace's equation are known. For the hyperbolic channel, it was decided to specify: symmetry about the x and y axis, uniform properties and zero velocity for $x \longrightarrow \pm\infty$, irrotational motion throughout, M at the center of the passage $x = y = 0$. For physical reasons, the solution of this problem is known to be unique. Notice that it is possible to specify, for example, the total flow through the passage in place of some of the aforementioned but that then the solution would not be unique.

4. With the boundary conditions in mind, guess values of the stream function ψ'. In the particular case of the hyperbolic channel shown in figure 8, M was chosen as 0.85 at the center. At this point, therefore, $\frac{\rho q}{\rho_o a_o}$ = 0.568 by computation figure 14. With δ chosen as 0.15, $\psi_o = 2\delta = 0.30$. Hence $\psi'(\xi = 0, \eta = 0.15) - \psi'(0.0) = \psi_\eta{}'\delta = \frac{\rho q}{\rho_o a_o} \frac{\delta}{q_i \psi_o} = \frac{0.568}{2} = 0.284$. To avoid continual use of small decimals, 1000 times this number is recorded in figure 6. The remaining ψ' values along $\xi = 0$ were set by using ψ' approximately proportional to ψ' for a solution already obtained for M center = 0.80. A good alternative procedure would have been to assign q/q_i constant along $\xi = 0$ (see equation (16)). Having an approximate ψ' on the boundary $\eta = 0.6$, it is constant for all ξ. For ξ large, ψ' is divided proportional to η. Finally, all remaining values of ψ' are put in by guess.

5. Compute the auxiliary quantities and the residuals by equation (20). The various values are arranged around the point, as in figure 7. Note that the $\psi_\xi{}'(\ln \rho/\rho_o)$ term has been omitted. This is possible only because it is of insignificant magnitude in the present case.

$$M \bullet \quad q_i$$

$$\{\psi'\} \left\{-\ln \frac{\rho}{\rho_o}\right\} \left\{\Delta\psi' + \psi'_\eta \left(\ln \frac{\rho}{\rho_o}\right)\eta = Q\right\}$$

$$\{q_i \quad \psi_\eta\} \left\{\frac{q}{a_o}\right\}$$

Figure 7

6. Relax by figure 2 to eliminate the residuals, Q. Periodically the error must be recomputed to take correctly into account the change of the $\psi_\eta'(\ln \rho/\rho_o)$ term not included in the relaxation pattern. When near the vertical tangent to the $\ln \rho/\rho_o - \dfrac{\rho q}{\rho_o a_o}$ curve (i.e. near $M = 1$), it is sometimes more convenient to change the $\ln \rho/\rho_o$ instead of ψ'.

7. Add more points where greater accuracy is required and recompute as above.

8. The required results are computed from the ψ' gradient values, equation (21), and the various computation curves. In regions near $M = 1$, the desired results may be more accurately determined from the values of $\ln \rho/\rho_o$. In the case of the hyperbolic channel, figure 8 shows the distribution of velocity. In the case of an airfoil, the most important results would be pressure distribution and lift. Computation figures 20 and 21 will supply the pressures. The lift can then be obtained by integration. Finally the lift, or pressure coefficient, can be found by using the value of $1/2\ \rho q^2$ from computation figure 22. The undisturbed stream dynamic head is generally used and therefore the effect of entropy on the $1/2\ \rho q^2$ against M relation has not been included in computation figure 22.

Treatment of Supersonic Flows, Especially Supersonic Regions in an Otherwise Subsonic Flow

As the speed of sound is approached by the fluid, the density-mass velocity relation approaches a vertical tangent as in computation figure 12. In this region the relaxation process is still able to yield a solution but the effect of changes in ψ' (or $ln\ \rho/\rho_o$) must be watched very closely so as to avoid making residuals worse rather than better.

The relaxation process, the removal of residuals by arbitrary changes of the dependent variable, becomes confusing for supersonic velocities. The following tentative method of solution has been found adequate for the problems solved to date.

The relation useful near the boundaries, equation (16), is approximately correct throughout the flow field and suggests working with q/q_i as variable in place of of $ln\,\rho$. As shown in appendix 4, equation (10) becomes

$$\psi_{\xi\xi} + \psi_\eta \left(ln\ \frac{q}{q_i} \right)_\eta - \psi_\eta \left(ln \left[1 + \frac{\psi_\xi{}^2}{\psi_\eta{}^2} \right]^{1/2} \right)_\eta - \psi_\xi\,(ln\rho)_\xi + \frac{2D^2\omega\rho}{q_i{}^2}\,\rho = 0 \qquad (22)$$

This equation would be no improvement over equation (10), except that the last three terms are generally very small. The first two terms then give

$$\frac{q}{q_i} = C(\xi)\ e^{-\int \frac{\psi_{\xi\xi}}{\psi_\eta} d\eta} \qquad (23)$$

If dimensionless variables are again introduced through

$$\psi = \rho_o\, a_o\, D\, \psi_o\, \psi' \quad \text{and} \quad q^* = \frac{q}{a_o q_i} \qquad (24)$$

there results

$$\psi_{\xi\xi}' + \psi_{\eta}' (\ln q^*)_{\eta} - 1/2\ \psi_{\eta}' \left\{ \ln \left[1 + \frac{\psi_{\xi}'^2}{\psi_{\eta}'^2} \right] \right\}_{\eta}$$

$$- \psi_{\xi}' (\ln \rho/\rho_o)_{\xi} + \frac{2D\ \omega\ \rho}{a_o \psi_o \rho_o q_i^2} = 0 \qquad (25)$$

and approximately

$$q^* = C(\xi) e^{-\int \frac{\psi_{\xi\xi}}{\psi_{\eta}} d\eta} \qquad (26)$$

A solution obtained with q^* constant, if such that the last three terms of equation (25) are really negligible, can be checked most easily by noting the value of $(\psi_{\eta}')_{\xi\xi}$ which is the residual in the equation (see appendix 4)

$$(\psi_{\eta}')_{\xi\xi} + \psi_{\eta}' (\ln q^*)_{\eta\eta} + (\psi_{\eta}')_{\eta} (\ln q^*)_{\eta} = 0 \qquad (27)$$

A solution is obtained in the following steps.

1. Lay out the problem as for a subsonic velocity solution following steps 1, 2, and 3.

2. On each ξ constant line choose a value of q^* = constant. By means of computation figure 14 determine the value of ψ_{η} at each net point (ψ_{ξ}^2 if not negligible can be estimated later and corrected for). Integrate ψ_{η} to find ψ and to check the boundary condition (when a streamline is given as at the surface of an airfoil or passage). If ψ does not satisfy the boundary condition, a new value of q^* is chosen and the computation repeated until it does.

3. The solution to this point has been obtained as a one-dimensional solution along velocity potential lines, each line being solved independent of the others. The residuals (of the ψ_{η}) can now be avaluated from equation (27) by computing $(\psi_{\eta}')_{\xi\xi}$ since the other two terms are zero.

4. Make adjustments to eliminate residuals. No definite instructions are given at this point because, to date, the residuals have been so very small that almost no adjustment has been required. Figure 9 shows the subsonic-supersonic transition in a hyperbolic nozzle obtained as outlined previously. This corresponds to the solution first obtained by a series expansion by Meyer (see reference 4).

Solutions with Compression Shocks

Since many practical gas dynamics problems start with gas of uniform properties and velocity, it is generally not necessary to consider variations in entropy. Thus all the computation curves with varying entropy are not needed. As soon as supersonic regions appear, discontinuities may occur in which the velocity drops and the pressure rises over an extremely short distance. Compression shock, as these phenomena are called, is well known in the literature (see, for example, reference 12). Compression shocks give rise to several effects not generally included in fluid mechanics solutions. In the first place, a compression shock involves the dissipation of mechanical energy resulting in an increase of entropy (see computation fig. 15). The entropy rise increases with an increase of the supersonic velocity of the fluid ahead of the shock measured relative to the shock. If the shock is oblique to the approaching stream, the normal component of velocity suffers a sudden change and hence the stream turns abruptly through some angle (see computation fig. 16). The entropy change through a stationary shock is thus dependent upon the initial stream Mach number and the shock obliquity.

In the stream following a shock the entropy remains constant along each streamline, as shown in appendix 1, but now the entropy is not constant throughout the region. Thus in the course of the numerical solution carried out exactly as indicated in the preceding sections, it is necessary to look up values on the computation curves at the entropy appropriate to the streamline passing through the particular point in question. Thus each time a computation curve must be used, the current value of ψ at that point must be observed and the value of the entropy appropriate to it must be used. As the solution progresses and the values of ψ at points change, correction for the attendant changes of the entropy must be made periodically.

This does not end the difficulties. If a shock wave is curved, or crosses a region of nonuniform (but irrotational) velocity, the velocity after the shock will not in general be distributed properly for the flow to be irrotational. Thus the rotation term on the right of equation (10) cannot be ignored in the flow following a shock wave. Quantitatively, the rotation following a shock is obtained by equation (9) from the change of entropy between streamlines, which in turn is obtained from computation figure 15 and the shock.

Note that along a given streamline the rotation is not constant but is proportional to the pressure (h_o is still constant everywhere by the assumptions of adiabatic, frictionless flow and uniform conditions at infinity). Thus in a region of flow following a shock, equation (20) becomes, using equation (9).

$$\psi_1' + \psi_2' + \psi_3' + \psi_4' - 4\psi_0' - \frac{(\psi_1' - \psi_3')(\ln \rho/\rho_{0\,1} - \ln \rho/\rho_{0\,3})}{4}$$

$$- \frac{(\psi_4' - \psi_2')(\ln \rho/\rho_{0\,4} - \ln \rho/\rho_{0\,2})}{4} + \frac{\delta^2}{\gamma q_1^2 \psi_0^2} \frac{\rho p}{\rho_0 p_0} \frac{\left(\frac{\Delta s}{R}\right)_4 - \left(\frac{\Delta s}{R}\right)_2}{\psi_4' - \psi_2'} = 0 \qquad (28)$$

where $\frac{d(s/R)}{d\psi}$ has been evaluated along a constant ξ line,

and $\frac{\rho p}{\rho_o p_o}$ is given on computation figures 23 and 24.

The final difficulty to be met is the fact that the shock is merely a "boundary condition" between a supersonic region and a subsonic region. Generally, given a solution with a shock, it is possible to extend the supersonic region beyond the position of the shock (at least a short distance) were it not present and to extend the subsonic region likewise. Thus the shock is a wave which moves back and forth until it has a magnitude which permits it to assume a steady, fixed location in the flow. The nature of the supersonic flow field "extended" and the subsonic flow field "extended" determines the stability of the shock wave. Experimentally shock waves are frequently found to waver or vibrate.

Actual solutions containing shocks are obtained in the following way:

1. A problem is solved as previously described, including regions of supersonic velocity.

2. A shock is arbitrarily placed in some location in the supersonic region. The more information, experimental or otherwise, about the probable location and shape of the shock wave, the better.

3. With this shock fixed the flow in the region following the shock is determined by the shock boundary conditions of stream function and entropy distribution.

4. On completing this solution by relaxation a check at the shock will generally show that the streamline direction following the shock does not agree with the shock obliquity assumed. The obliquity is changed to get agreement of direction of the streamlines and step 3 is repeated.

5. A few repetitions suffice to get a sufficiently accurate solution.

A solution with a shock in the hyperbolic passage obtained in this manner is shown in figure 10.

CONCLUSIONS

Numerical methods for obtaining solutions of the two-dimensional, adiabatic flow of frictionless, perfect gases is described in detail and illustrated by solutions of the flow of air through a hyperbolic passage at widely varying velocities.

The relaxation method applied to general passages or airfoil shapes can readily supply all data desired for the flow of incompressible fluids. These solutions can be corrected for compressibility effects up to the appearance of supersonic regions by use of the same method.

After supersonic regions appear, other methods described permit the further correction of the flow for these effects. Finally, solutions with shocks, including all of the attendant rotation and entropy change effects, are obtained by a step-by-step process.

All of the methods described have one enormous advantage over analytical methods of solution of these problems. They permit the computer to use all of the facts he knows about the phenomena throughout the computations.

Many curves presenting the properties of air as required for these computations are appended.

Harvard University,
Cambridge, Mass., March 1, 1944.

APPENDIX I

ROTATIONAL MOTION

For most fluid mechanics work, the equation of motion of the fluid can be replaced by the fact that the velocity distribution is irrotational. For the supersonic flow of compressible fluids in which shock waves occur, the velocity distribution will not be irrotational.

Consider a general case of the motion of a compressible, frictionless, perfect gas between the curved streamlines of figure 11. The fluid element rotates about an axis normal to the paper at rate given by

$$2\omega = -\frac{\partial q}{\partial n} - \frac{q}{r} \qquad (29)$$

The pressure gradient normal to the streamlines must produce the centripetal acceleration of the fluid element; thus

$$\frac{\partial p}{\partial n} = \frac{\rho q^2}{r} \qquad (30)$$

Eliminate r between (29) and (30)

$$2\omega = -\frac{\partial q}{\partial n} - \frac{1}{\rho q}\frac{\partial p}{\partial n} \qquad (31)$$

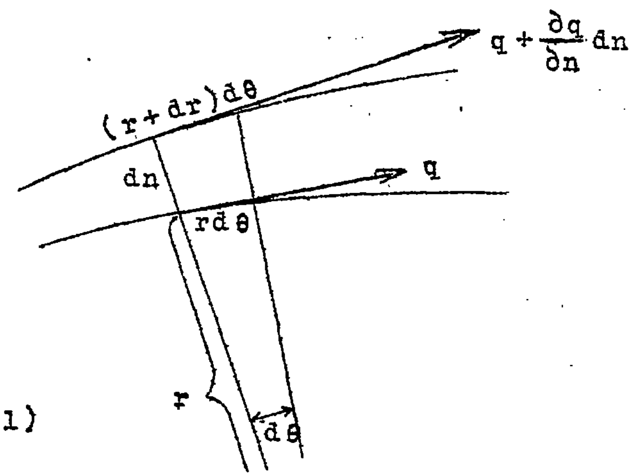

Figure 11

Introduce the stagnation enthalpy

$$h_o = h + \frac{q^2}{2} \tag{32}$$

and the thermodynamic relation

$$dh = T\,ds + \frac{dp}{\rho} \tag{33}$$

and equation (31) for the rotation of the fluid element becomes

$$2\omega = \frac{1}{q}\left(T\,\frac{\partial s}{\partial n} - \frac{\partial h_o}{\partial n}\right) \tag{34}$$

Now introduce the stream function $d\psi = \rho q\,dn$ and the gas law $p = \rho R T$

$$2\omega = \frac{p}{R}\;\frac{\partial s}{\partial \psi} - \rho\,\frac{\partial h_o}{\partial \psi} \tag{35}$$

The condition for irrotational motion of a perfect gas is seen to be constant entropy and constant stagnation enthalpy throughout the region, or else a delicate balance between s and h_o. The balanced case is important since it insures zero rotation for the adiabatic flow of a gas from a large region of zero velocity even though the temperature is not uniform there. The case of greatest importance is the adiabatic flow from uniform, zero velocity conditions at infinity. For this case h_o is constant everywhere and s is constant up to the first shock wave. Thus the flow is irrotational up to the first shock wave and rotational thereafter.

To find the distribution of rotation behind the shock wave, differentiate equation (32) along a streamline, observing that for adiabatic flow h_o is constant everywhere,

$$\frac{\partial h}{\partial L} + q\,\frac{\partial q}{\partial L} = 0 \tag{36}$$

Also by the Bernoulli equation

$$\frac{1}{\rho}\frac{\partial p}{\partial L} + q\frac{\partial q}{\partial L} = 0 \qquad (37)$$

According to equation (33)

$$\frac{\partial s}{\partial L} = \frac{1}{T}\left(\frac{\partial h}{\partial L} - \frac{1}{\rho}\frac{\partial p}{\partial L}\right) = 0 \qquad (38)$$

The entropy remains constant along streamlines between shocks. Thus the rotation of the fluid is proportional to the pressure along streamlines in regions between shocks. The proportionality constant varies from streamline to streamline according to the distribution of entropy between streamlines produced by the shock wave.

APPENDIX II

THE COMPUTATION CURVES

Most of the curves found useful in computation follow from these well-known thermodynamic relations for a constant specific heat, perfect gas.

$$p = \rho R T \qquad (a)$$

$$h = c_p T = \frac{a^2}{\gamma - 1} \qquad (b) \qquad (39)$$

$$ds = c_p \frac{dT}{T} - R\frac{dp}{p} = c_v \frac{dT}{T} - R\frac{d\rho}{\rho} = c_v \frac{dp}{p} - c_p \frac{d\rho}{\rho} \qquad (c)$$

The conservation of energy for the adiabatic flow of a frictionless fluid is

$$h + \frac{q^2}{2} = h_o\text{, a constant along a streamline} \qquad (40)$$

Introduce the acoustic velocity from (39b)

$$\frac{h}{h_o} = \frac{T}{T_o} = \frac{a^2}{a_o^2} = 1 - \frac{\gamma - 1}{2}\left(\frac{q}{a_o}\right)^2 \tag{41}$$

By integration of equation (39c)

$$\frac{\rho}{\rho_o} = e^{-\frac{s-s_o}{R}}\left(\frac{T}{T_o}\right)^{\frac{1}{\gamma-1}} = e^{-\frac{s-s_o}{R}}\left\{1 - \frac{\gamma-1}{2}\left(\frac{q}{a_o}\right)^2\right\}^{\frac{1}{\gamma-1}} \tag{42}$$

Computation figures 12, 13, and 14 follow immediately.

Equation (41) can be rewritten as

$$\frac{1}{M^2} = \frac{1}{(q/a_o)^2} - \frac{\gamma-1}{2} \tag{43}$$

This relation is independent of the entropy changes and is given as computation figure 19. This with computation figures 12, 13, and 14 yields figures 17 and 18. Again by (39c)

$$\frac{p}{p_o} = e^{-\frac{s-s_o}{R}}\left(\frac{T}{T_o}\right)^{\frac{\gamma}{\gamma-1}} = e^{-\frac{s-s_o}{R}}\left\{1 - \frac{\gamma-1}{2}\left(\frac{q}{a_o}\right)^2\right\}^{\frac{\gamma}{\gamma-1}} \tag{44}$$

This equation, together with (42), permits the construction of computation figures 20, 21, 23, and 24.

A general relation for the dynamic pressure is obtained from equations (42) and (43).

$$\frac{\rho q^2}{p_o} = \frac{\gamma M^2 e^{-\frac{s-s_o}{R}}}{\left\{1 + \frac{\gamma-1}{2}M^2\right\}^{\frac{\gamma}{\gamma-1}}} \tag{45}$$

Since the dynamic pressure is used only to determine pressure coefficients, the computation figure 22 is plotted for only $s-s_o = 0$.

The computation figure 15 and 16 for compression shocks follow from Prandtl's equation for normal shocks

$$q_b \, q_a = q_{cr}^2 \tag{46}$$

where

q_b velocity before the shock

q_a velocity after the shock

q_{cr} critical velocity $= \left\{ \frac{2}{\gamma + 1} a^2 + \frac{\gamma - 1}{\gamma + 1} q_s^2 \right\}^{1/2}$

together with equations (39).

The entropy increase through a normal shock was computed from

$$\frac{s - s_0}{R} = \frac{1}{\gamma - 1} \left\{ \ln \left(\frac{2\gamma M^2}{\gamma + 1} - \frac{\gamma - 1}{\gamma + 1} \right) + \gamma \ln \left(\frac{2}{(\gamma + 1) M^2} + \frac{\gamma - 1}{\gamma + 1} \right) \right\} \tag{47}$$

The fact that an oblique shock is a normal shock to a component of the velocity completes the information required for computation figures 15 and 16.

APPENDIX III

The transformation of the equation for the stream function from the physical x,y plane to a new ξ, η plane gives a simple result for a conformal transformation. A conformal transformation results for ξ and η such that

$$\eta_Y = \xi_X , \quad \eta_X = - \xi_Y \tag{2}$$

This is equation (2) of the report which defines the stream function and velocity potential of an irrotational flow of an incompressible fluid except that, for convenience,

$$X = \frac{x}{D} , \; Y = \frac{y}{D} \tag{48}$$

First it is shown that

$$\Delta_{xy} \; \cancel{\Delta_{XY}} \varphi = \frac{q_1^2}{D^2} \Delta_{\xi\eta} \varphi \tag{49}$$

for any function $\cancel{\varphi(X,Y)}$. $\varphi(x,y)$

A straightforward derivation seems in this case to be simplest:

$$\varphi_X = \varphi_\xi \, \xi_X + \varphi_\eta \, \eta_X \tag{50}$$

Repeating the process and rearranging terms,

$$\varphi_{XX} = \varphi_{\xi\xi} \, \xi_X^2 + \varphi_{\eta\eta} \, \eta_X^2 + 2 \varphi_{\xi\eta} \, \eta_X \xi_X + \varphi_\xi \, \xi_{XX} + \varphi_\eta \, \eta_{XX} \tag{51}$$

A similar expression for φ_{YY} when added to equation (51) gives

$$\varphi_{XX} + \varphi_{YY} = \varphi_{\xi\xi} \left(\xi_X^2 + \xi_Y^2\right) + \varphi_{\eta\eta} \left(\eta_X^2 + \eta_Y^2\right)$$

$$+ 2 \varphi_{\eta\xi} \left(\eta_X \xi_X + \eta_Y \xi_Y\right) + \varphi_\xi \left(\xi_{XX} + \xi_{YY}\right) + \varphi_\eta \left(\eta_{XX} + \eta_{YY}\right) \tag{52}$$

By equation (2) this reduces to

$$\Delta_{XY} \varphi \equiv \varphi_{XX} + \varphi_{YY} = q_1^2 \left(\varphi_{\xi\xi} + \varphi_{\eta\eta}\right) \equiv q_1^2 \Delta_{\xi\eta} \varphi \tag{53}$$

where

$$q_1^2 = \xi_X^2 + \xi_Y^2 = \eta_X^2 + \eta_Y^2 \tag{54}$$

Returning to the physical coordinates $x = XD$, $y = YD$, equation (~~48~~ 49) follows.

Now the conversion of the differential equation (10a) for the stream function, ψ, from x,y to $\xi\eta$ coordinates follows immediately from the identity

$$\frac{\partial}{\partial x}\left(\frac{1}{\rho} \frac{\partial \psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{1}{\rho} \frac{\partial \psi}{\partial y}\right) = \frac{1}{2} \left\{ \Delta \frac{\psi}{\rho} + \frac{\Delta \psi}{\rho} - \psi \Delta \frac{1}{\rho} \right\} \tag{55}$$

Thus equation (10a)

$$\frac{\partial}{\partial x}\left(\frac{1}{\rho}\ \frac{\partial\psi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{1}{\rho}\ \frac{\partial\psi}{\partial y}\right)=-2\omega \qquad (10a)$$

becomes

$$\frac{\partial}{\partial\xi}\left(\frac{1}{\rho}\ \frac{\partial\psi}{\partial\xi}\right)+\frac{\partial}{\partial\eta}\left(\frac{1}{\rho}\ \frac{\partial\psi}{\partial\eta}\right)=-\frac{2\,D^2\omega}{q_i^{\,2}} \qquad (14a)$$

APPENDIX IV

DERIVATION OF EQUATION USED TO COMPUTE SUPERSONIC VELOCITY REGIONS

To get an equation for the stream function ψ with $\frac{q}{q_i}$ as coefficient instead of $ln\,\rho$, begin with equation (14b).

$$\psi_{\xi\xi}+\psi_{\eta\eta}=\psi_\xi\ (ln\,\rho)_\xi-\psi_\eta\ (ln\,\rho)_\eta+\frac{2\,D^2\omega\rho}{q_i^{\,2}}=0 \qquad (14b)$$

By equation (15)

$$\frac{q}{q_i}=\frac{\psi_\eta}{D\rho}\left(\frac{\psi_\xi^{\,2}}{\psi_\eta^{\,2}}+1\right)^{1/2} \qquad (56)$$

where the square root is (for numerical computations) no trouble, as its value is generally very near unity.

Rewrite equation (14b)

$$\psi_{\xi\xi}+\psi_\eta\left(ln\,\frac{\psi_\eta}{\rho}\right)_\eta-\psi_\xi\ (ln\,\rho)_\xi+\frac{2\,D^2\omega\rho}{q_i^{\,2}}=0 \qquad (57)$$

and substitute from equation (56)

$$\psi_{\xi\xi}+\psi_\eta\left(ln\,\frac{q}{q_i}\right)_\eta-\frac{1}{2}\ \psi_\eta\left\{ln\left(1+\frac{\psi_\xi^{\,2}}{\psi_\eta^{\,2}}\right)\right\}_\eta$$

$$-\psi_\xi\,(ln\,\rho)_\xi+\frac{2\,D^2\omega\rho}{q_i^{\,2}}=0 \qquad (22)$$

A sometimes more useful form of equation (22) follows by noting that the last three terms are generally very small and that the velocity q is related to the ψ gradient by the computation curves. Differentiate the first two terms of equation (22) and neglect the last three terms.

$$(\psi_\eta)_{\xi\xi} + (\psi_\eta)_\eta \left(\ln \frac{q}{q_i}\right)_\eta + \psi_\eta \left(\ln \frac{q}{q_i}\right)_{\eta\eta} = 0 \qquad (58)$$

This becomes equation (27), if dimensionless variables are introduced. In the use of equation (58), the gradient is taken as ψ_η the ψ_ξ term being very small.

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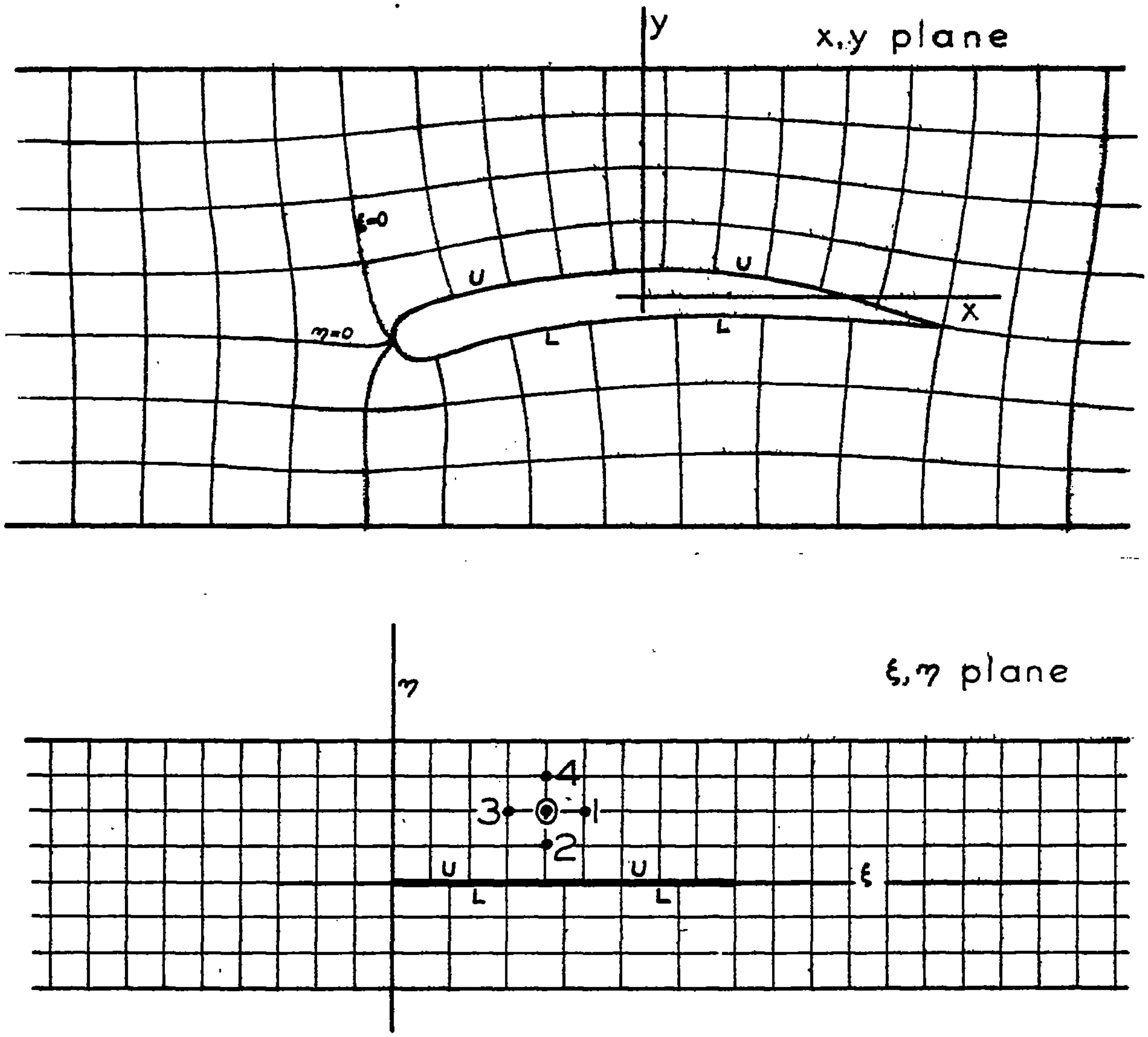
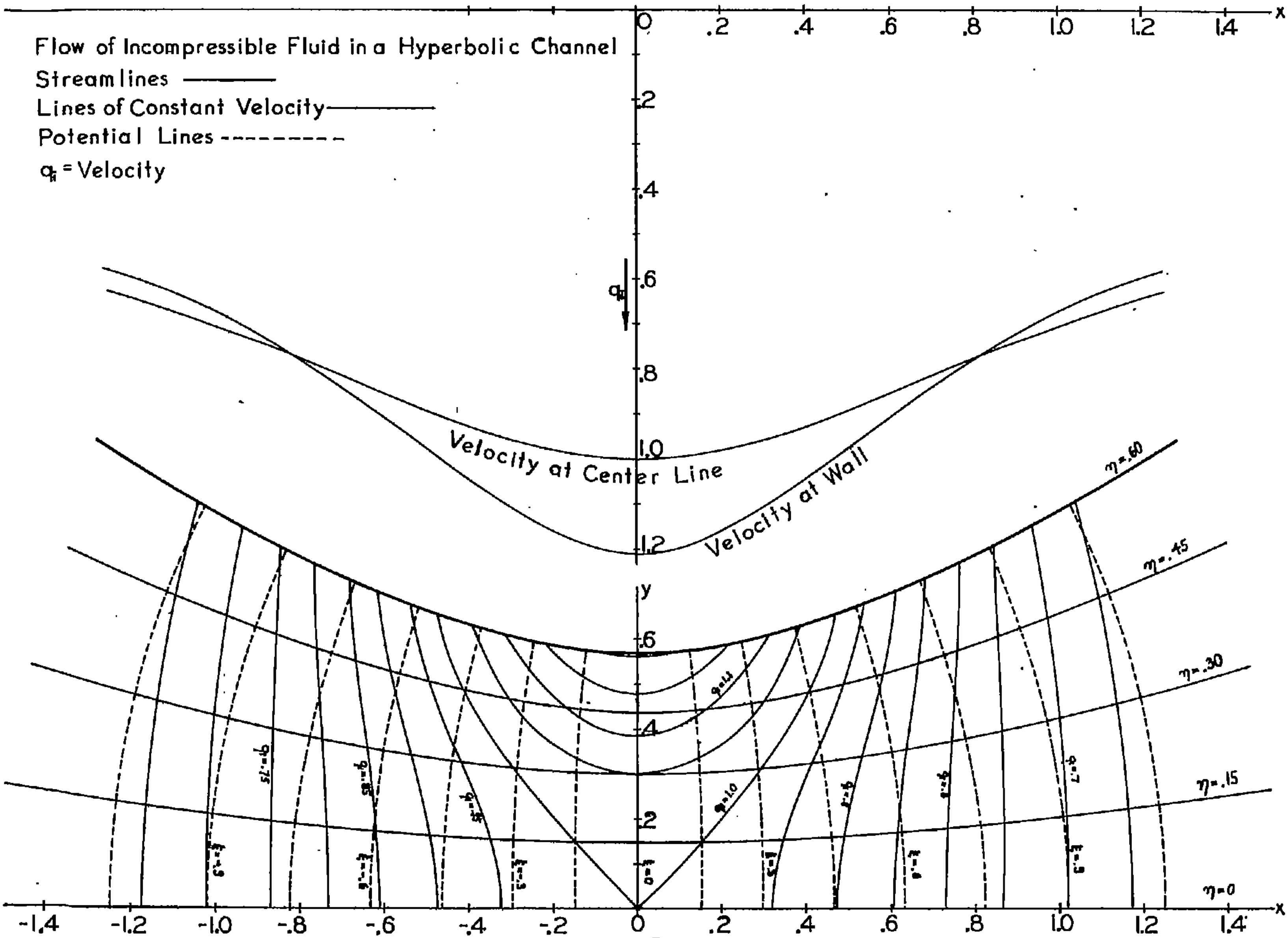


Fig. 4.



Flow of Incompressible Fluid in a Hyperbolic Channel
Streamlines
Lines of Constant Velocity
Potential Lines
q_1 = Velocity
Velocity at Center Line
Velocity at Wall
η=.60
η=.45
η=.30
η=.15
η=0
x
y

Fig.5.

 Symmetrical Flow in Hyperbolic Channel

Mach Number at Center of Passage M=.835

$\boxed{\psi'}$ Final Value of Stream Function

(Key on Figure 6b)

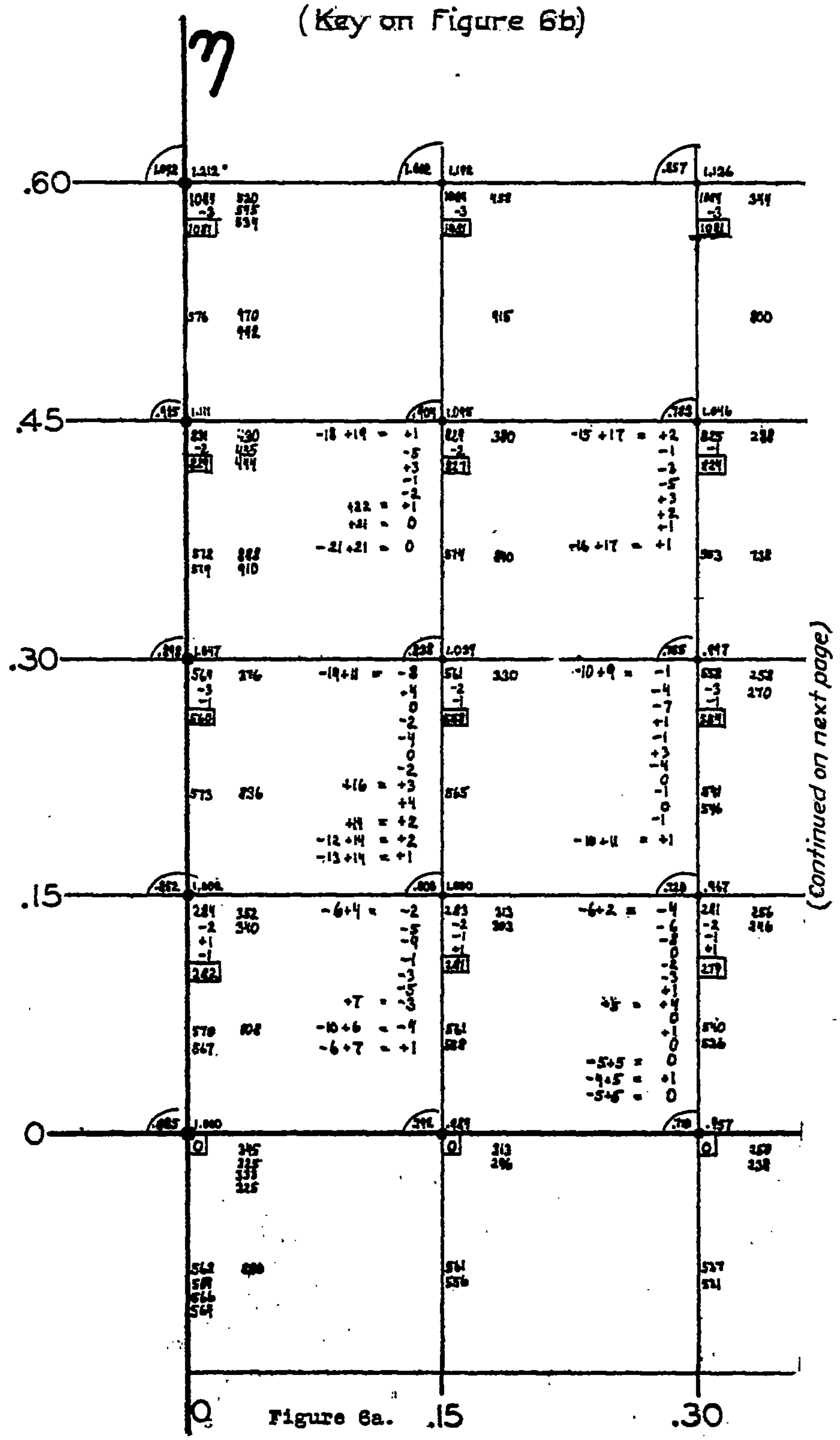
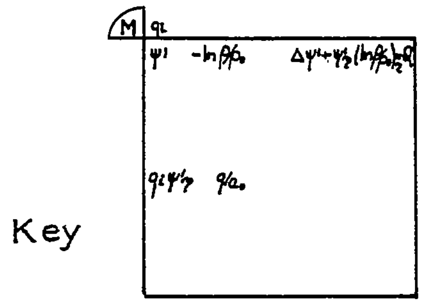
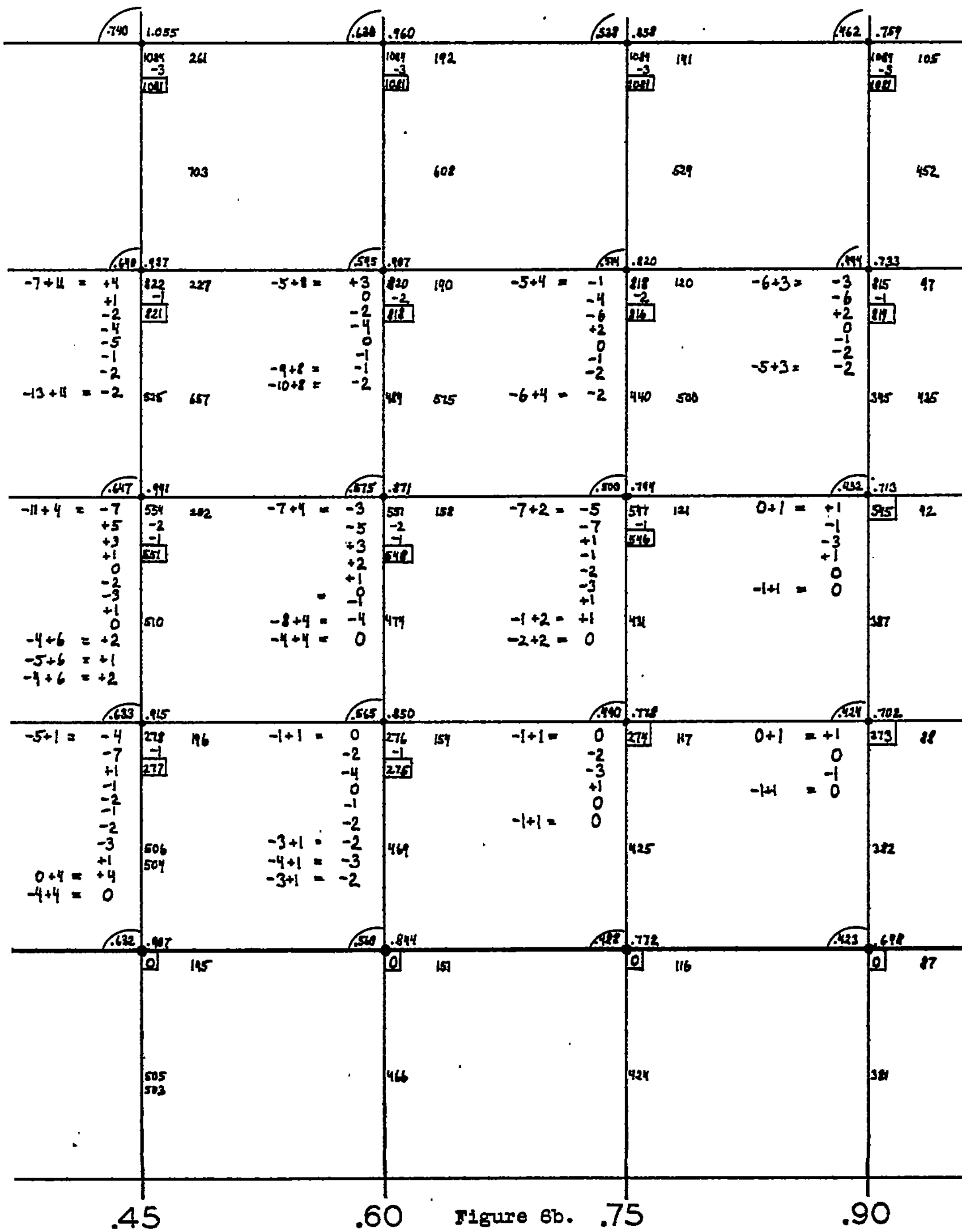


Figure 6a.





(Continued on next page)

Figure 6b.

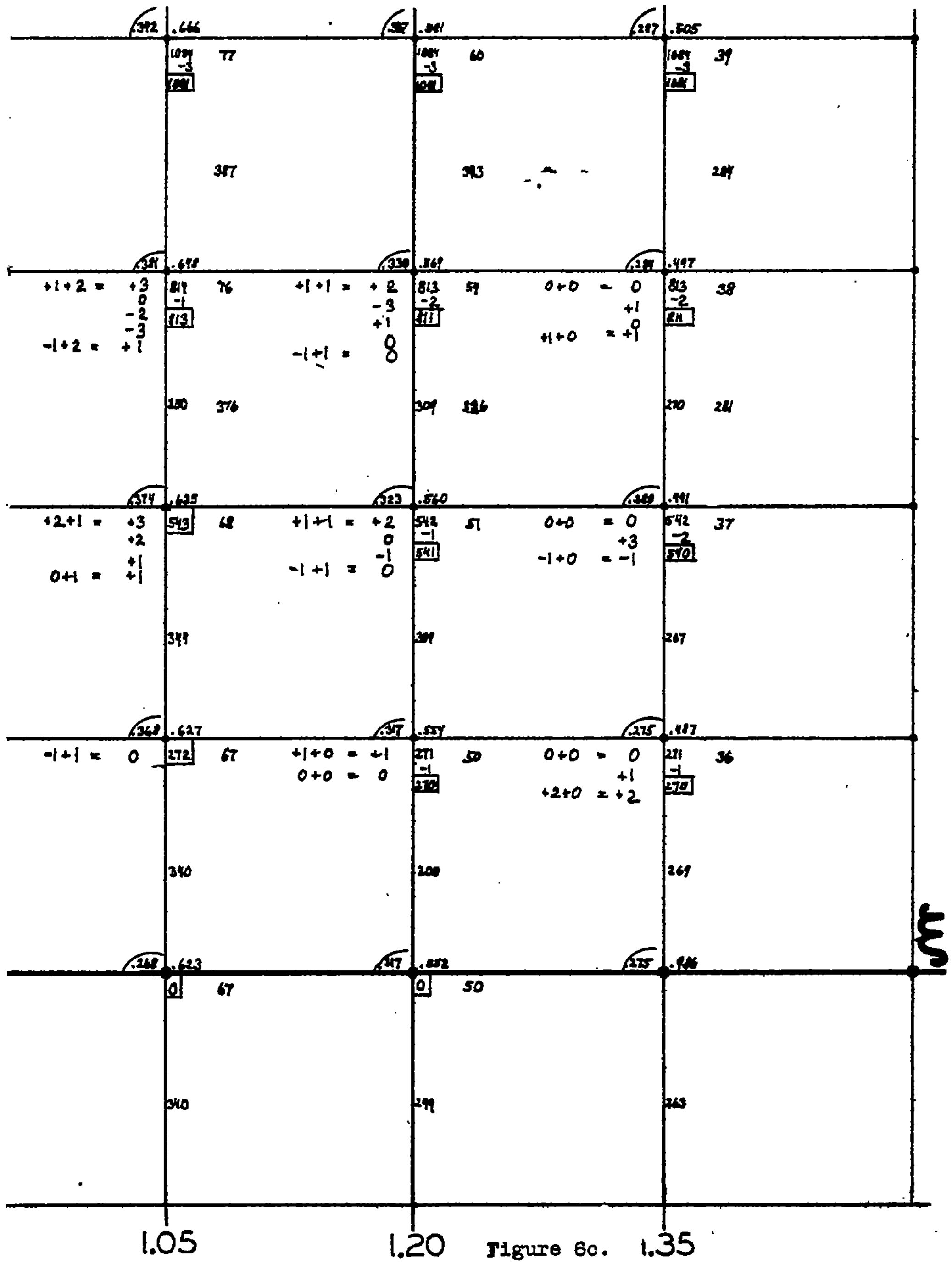


Figure 6c.

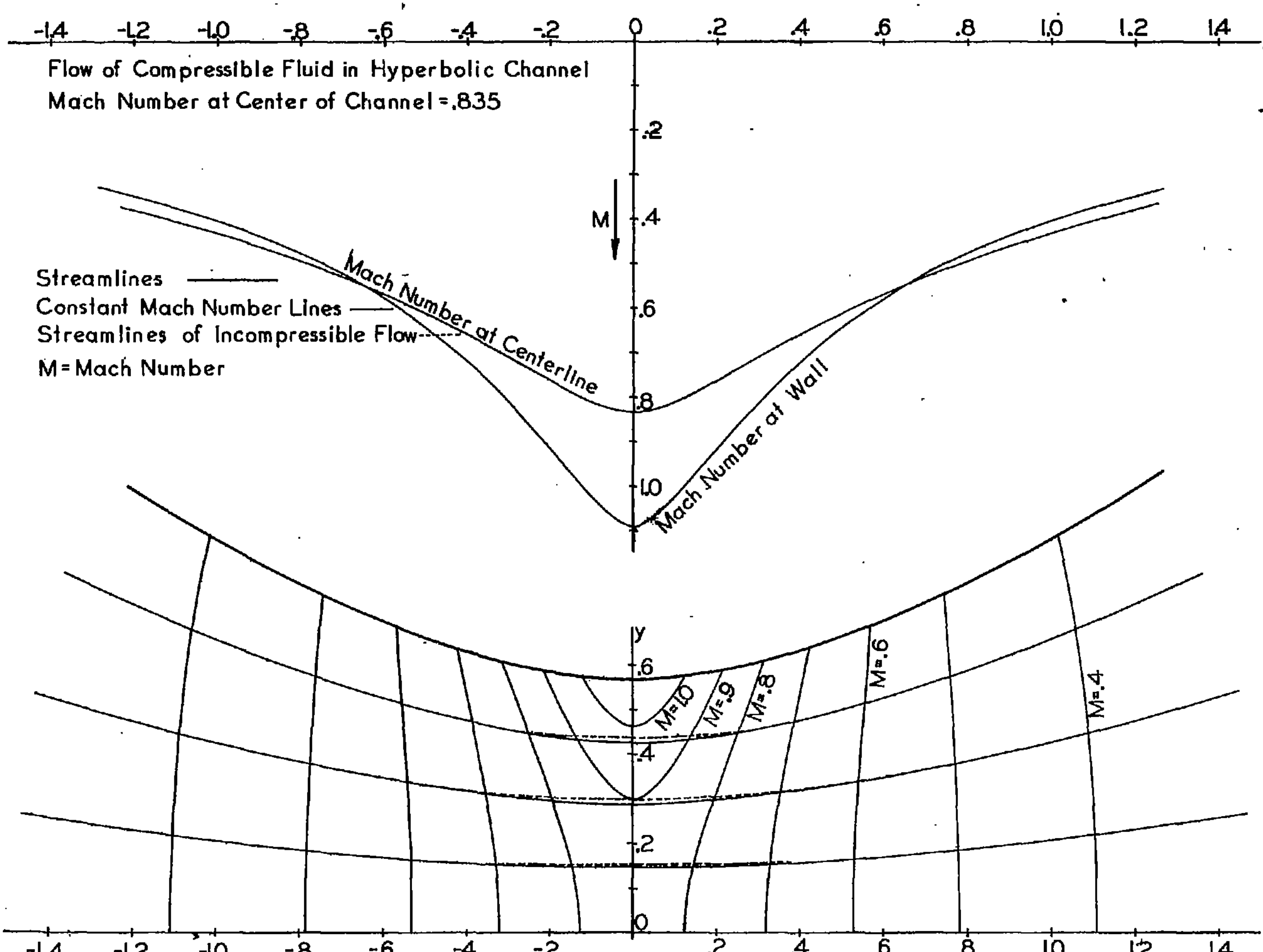

Flow of Compressible Fluid in Hyperbolic Channel
Mach Number at Center of Channel = .835
Streamlines
Constant Mach Number Lines
Streamlines of Incompressible Flow
M = Mach Number
Mach Number at Centerline
Mach Number at Wall
M
x
y
M=1.0
M=.9
M=.8
M=.6
M=.4
-1.4
-1.2
-1.0
-.8
-.6
-.4
-.2
0
.2
.4
.6
.8
1.0
1.2
1.4

Fig.8

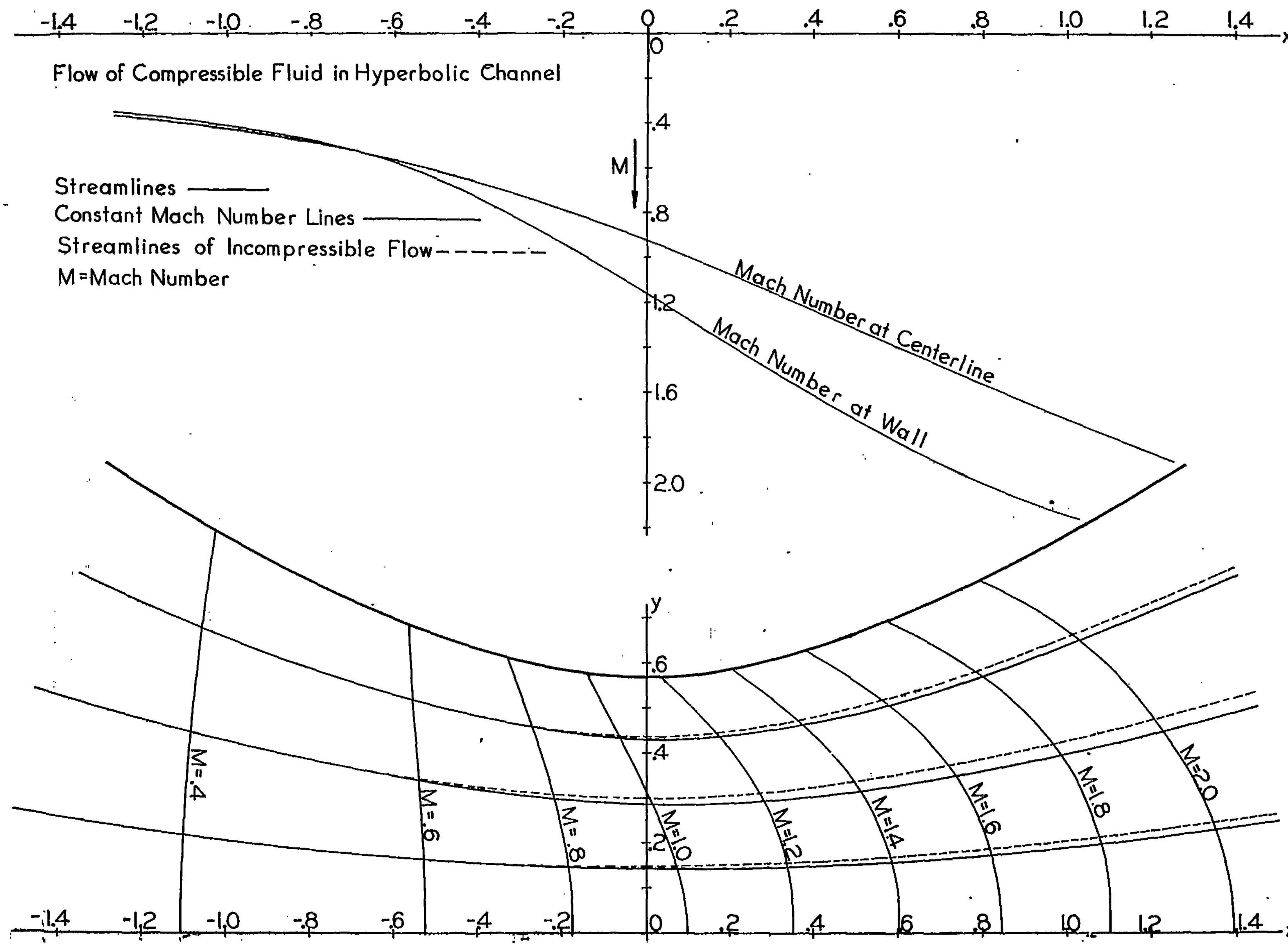

Flow of Compressible Fluid in Hyperbolic Channel
Streamlines
Constant Mach Number Lines
Streamlines of Incompressible Flow
M=Mach Number
M
Mach Number at Centerline
Mach Number at Wall
M=.4
M=.6
M=.8
M=1.0
M=1.2
M=1.4
M=1.6
M=1.8
M=2.0
x
y

Fig.9.

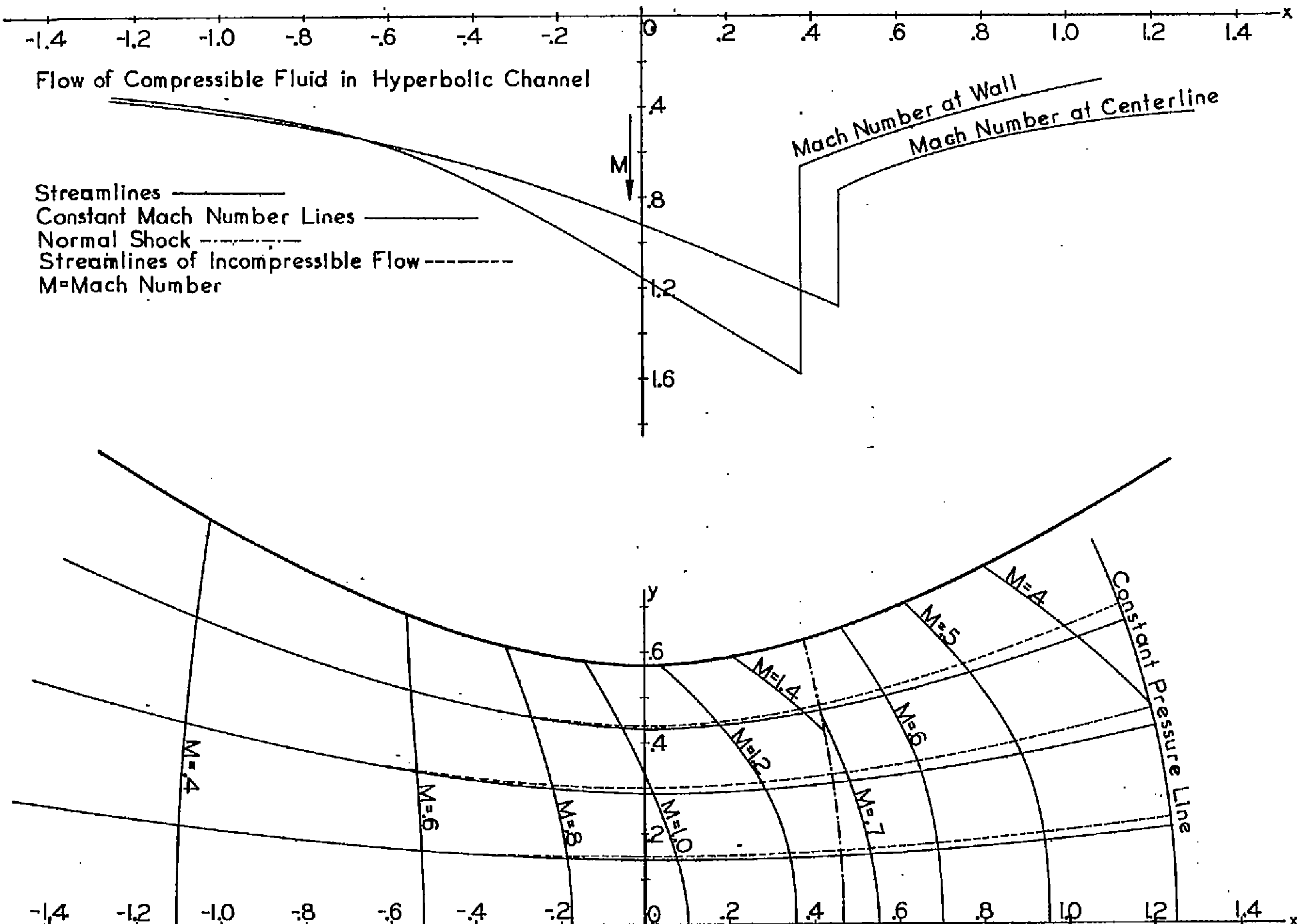

Flow of Compressible Fluid in Hyperbolic Channel
Streamlines
Constant Mach Number Lines
Normal Shock
Streamlines of Incompressible Flow
M=Mach Number
Mach Number at Wall
Mach Number at Centerline
M
Constant Pressure Line
M=.4
M=.6
M=.8
M=1.0
M=1.2
M=1.4
M=.7
M=.6
M=.5
M=.4
x
y

Fig. 10

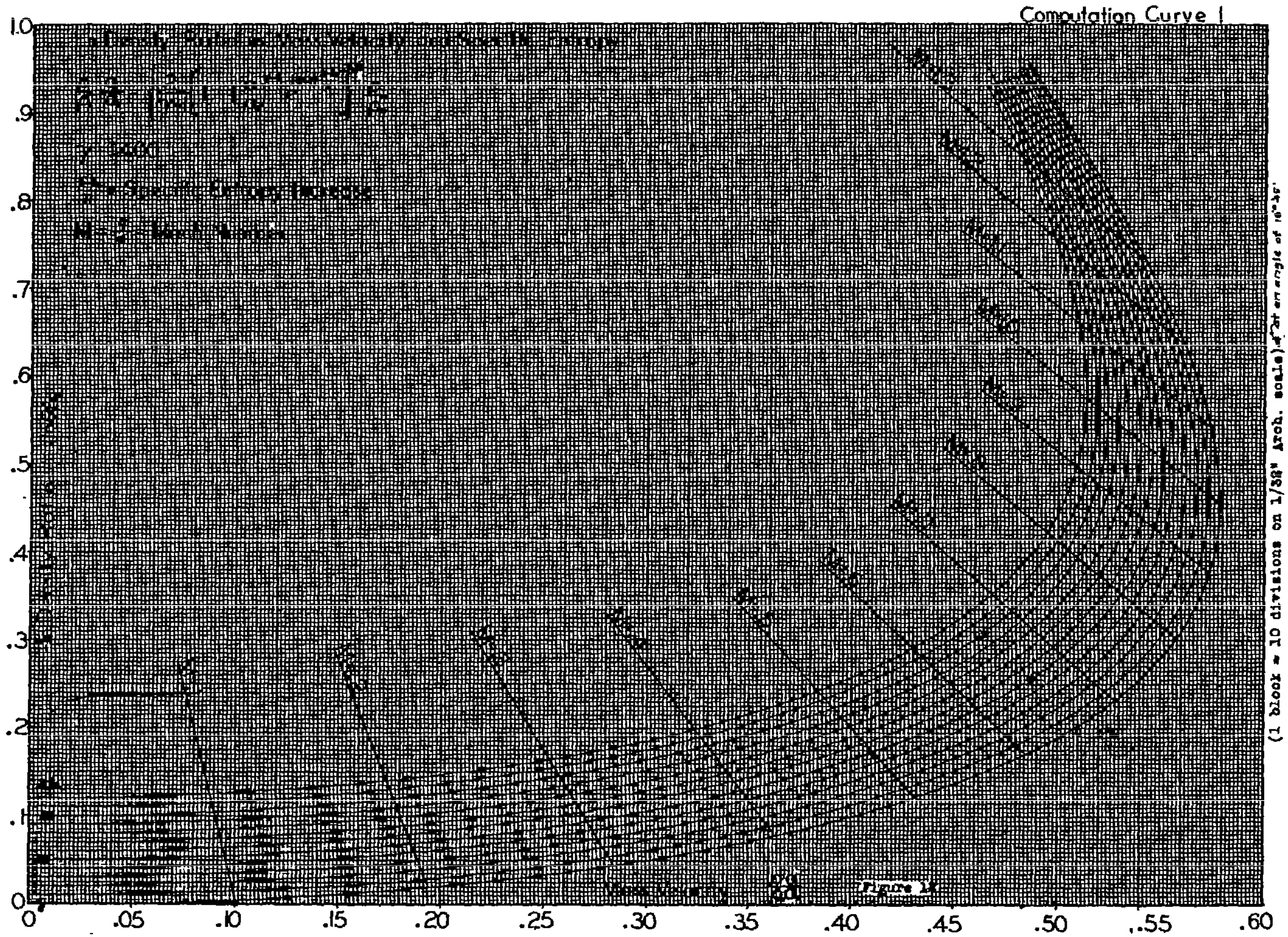
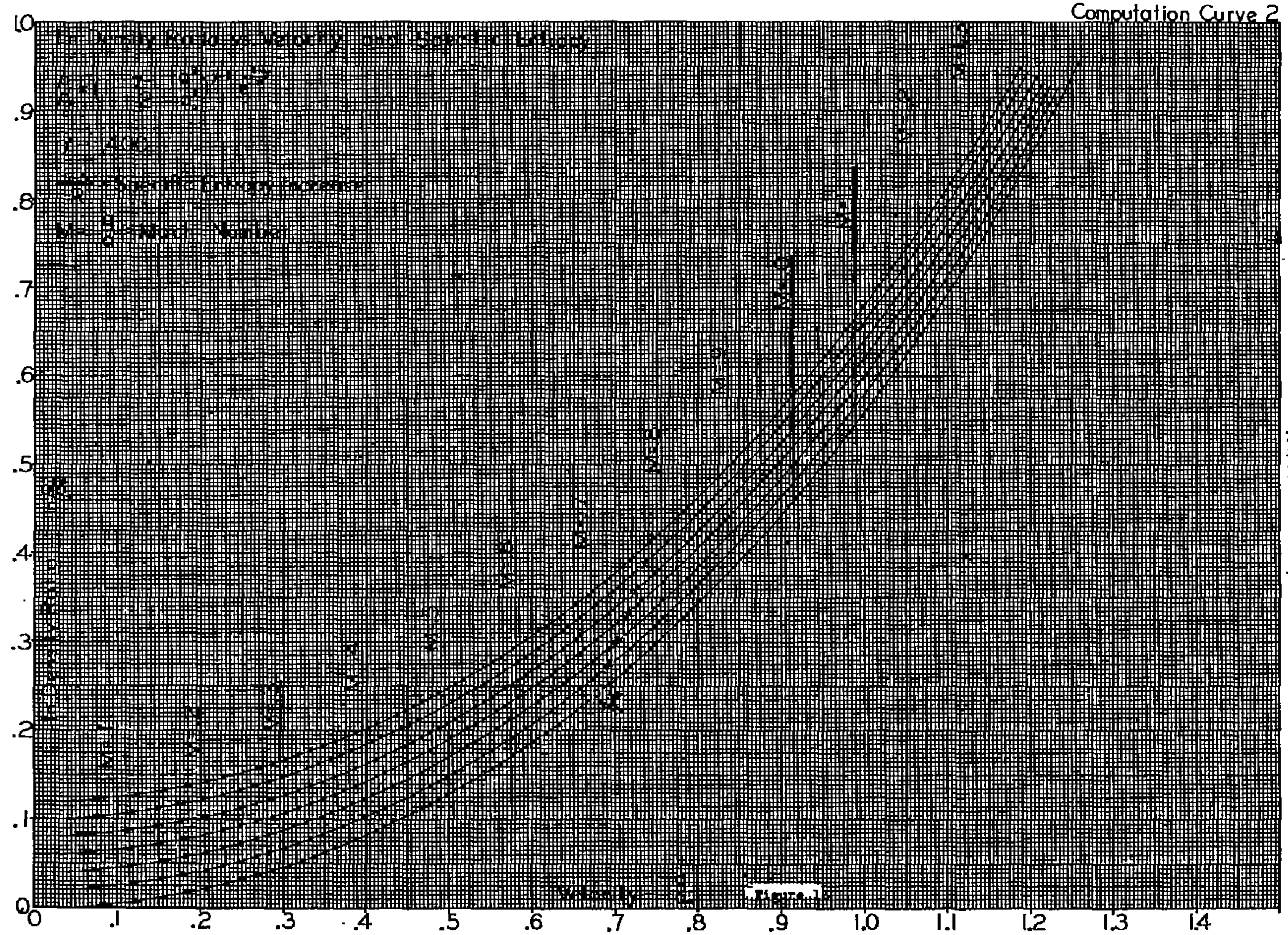

Computation Curve I
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
0
.05
.10
.15
.20
.25
.30
.35
.40
.45
.50
.55
.60
(1 block = 10 divisions on 1/32" Arch. scale)

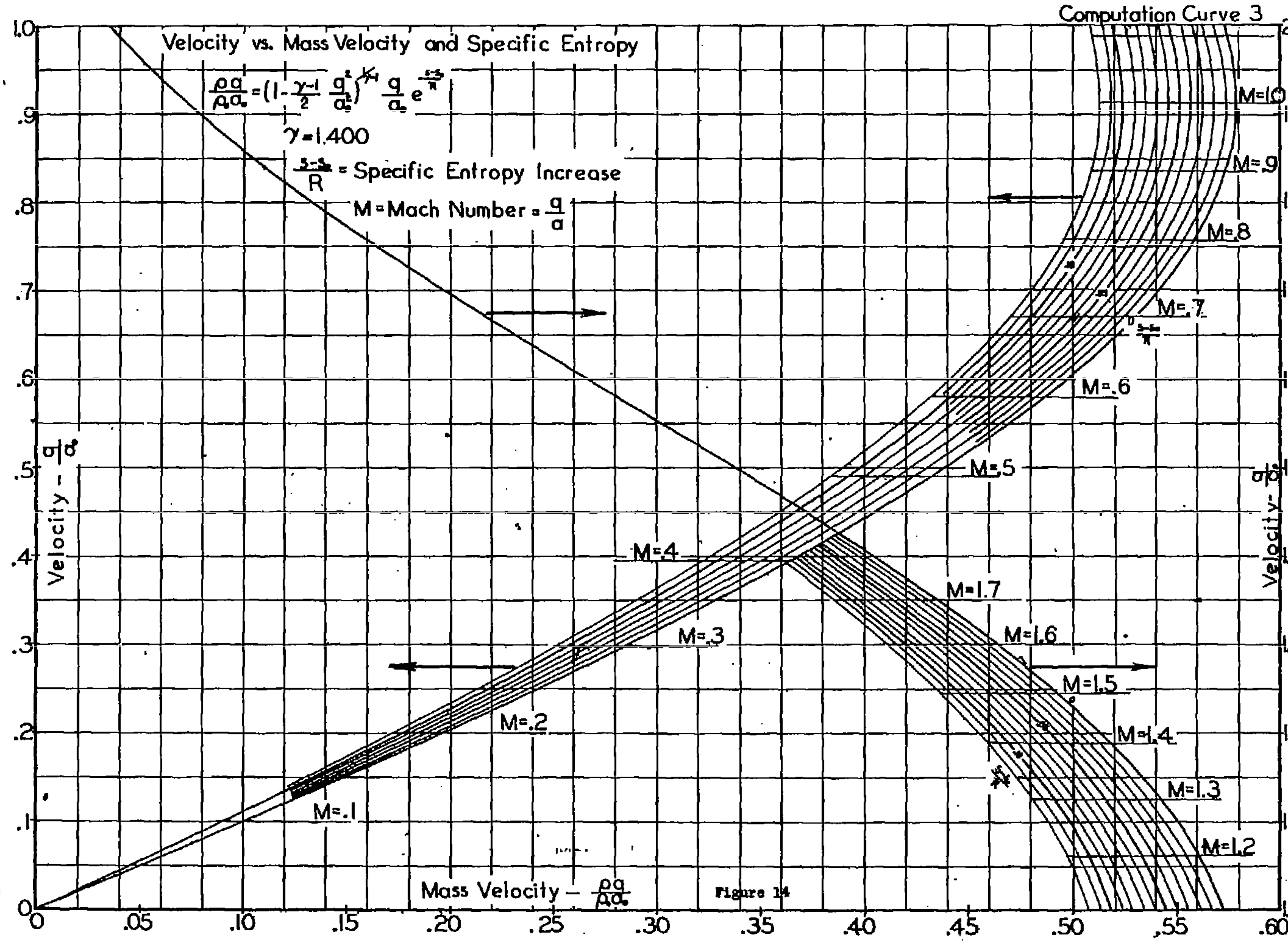
Fig. 18

Computation Curve 2



(1 block = 10/32)

Fig. 13


Computation Curve 3
Velocity vs. Mass Velocity and Specific Entropy
$\frac{\rho q}{\rho_0 a_0} = \left(1 - \frac{\gamma-1}{2}\frac{q^2}{a_0^2}\right)^{\frac{1}{\gamma-1}} \frac{q}{a_0} e^{-\frac{s-s_0}{R}}$
$\gamma = 1.400$
$\frac{s-s_0}{R}$ = Specific Entropy Increase
M = Mach Number = $\frac{q}{a}$
M=.1
M=.2
M=.3
M=.4
M=.5
M=.6
M=.7
M=.8
M=.9
M=1.0
M=1.2
M=1.3
M=1.4
M=1.5
M=1.6
M=1.7
Velocity – $\frac{q}{a_0}$
Mass Velocity – $\frac{\rho q}{\rho_0 a_0}$
Figure 14

(1 block = 10/38)

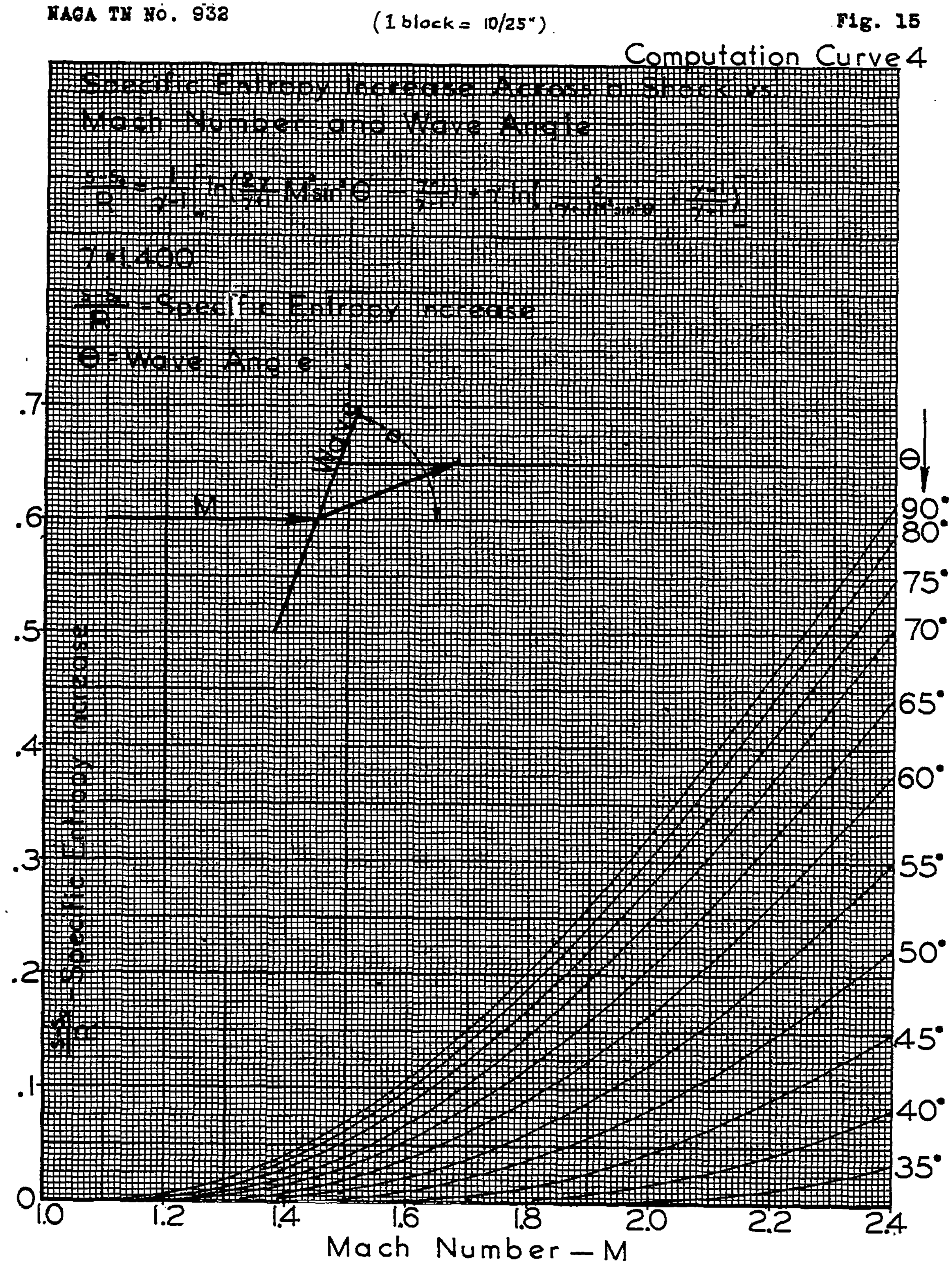

(1 block = 10/25")
Computation Curve 4
Specific Entropy Increase Across a Shock vs Mach Number and Wave Angle
γ=1.400
-Specific Entropy Increase
θ=Wave Angle
Wave
M
θ
90°
80°
75°
70°
65°
60°
55°
50°
45°
40°
35°
.7
.6
.5
.4
.3
.2
.1
0
1.0
1.2
1.4
1.6
1.8
2.0
2.2
2.4
Mach Number — M
Specific Entropy Increase

Figure 15

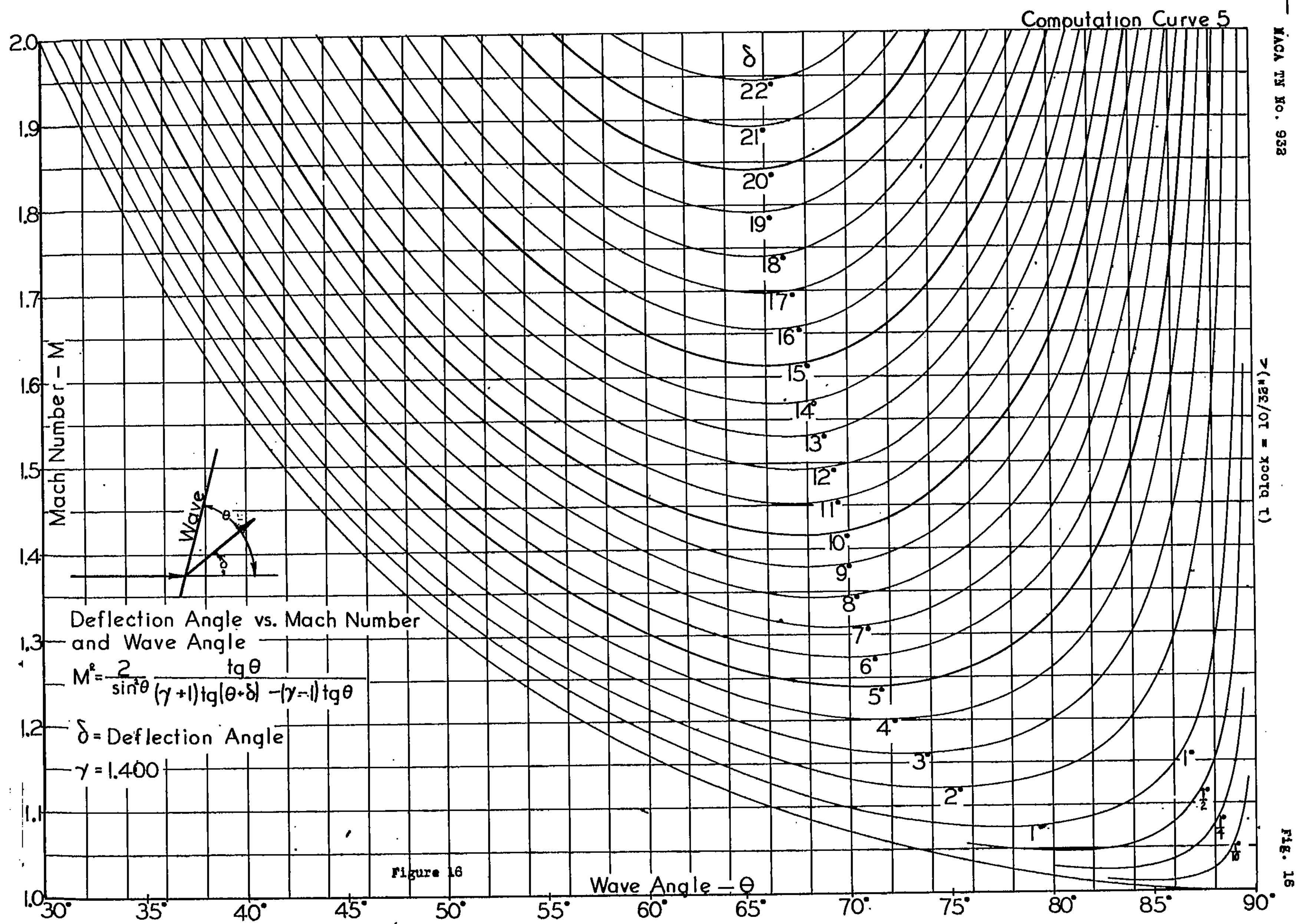

Computation Curve 5
δ
22°
21°
20°
19°
18°
17°
16°
15°
14°
13°
12°
11°
10°
9°
8°
7°
6°
5°
4°
3°
2°
1°
1/2°
1/4°
Mach Number – M
Wave Angle – θ
Wave
θ
δ
Deflection Angle vs. Mach Number and Wave Angle
$M^2 = \frac{2}{\sin^2\theta} \frac{tg\,\theta}{(\gamma+1)\,tg(\theta+\delta) - (\gamma-1)\,tg\,\theta}$
δ = Deflection Angle
γ = 1.400
(1 block = 10/32")
2.0
1.9
1.8
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1.0
30°
35°
40°
45°
50°
55°
60°
65°
70°
75°
80°
85°
90°

Figure 16

NACA TN No. 938

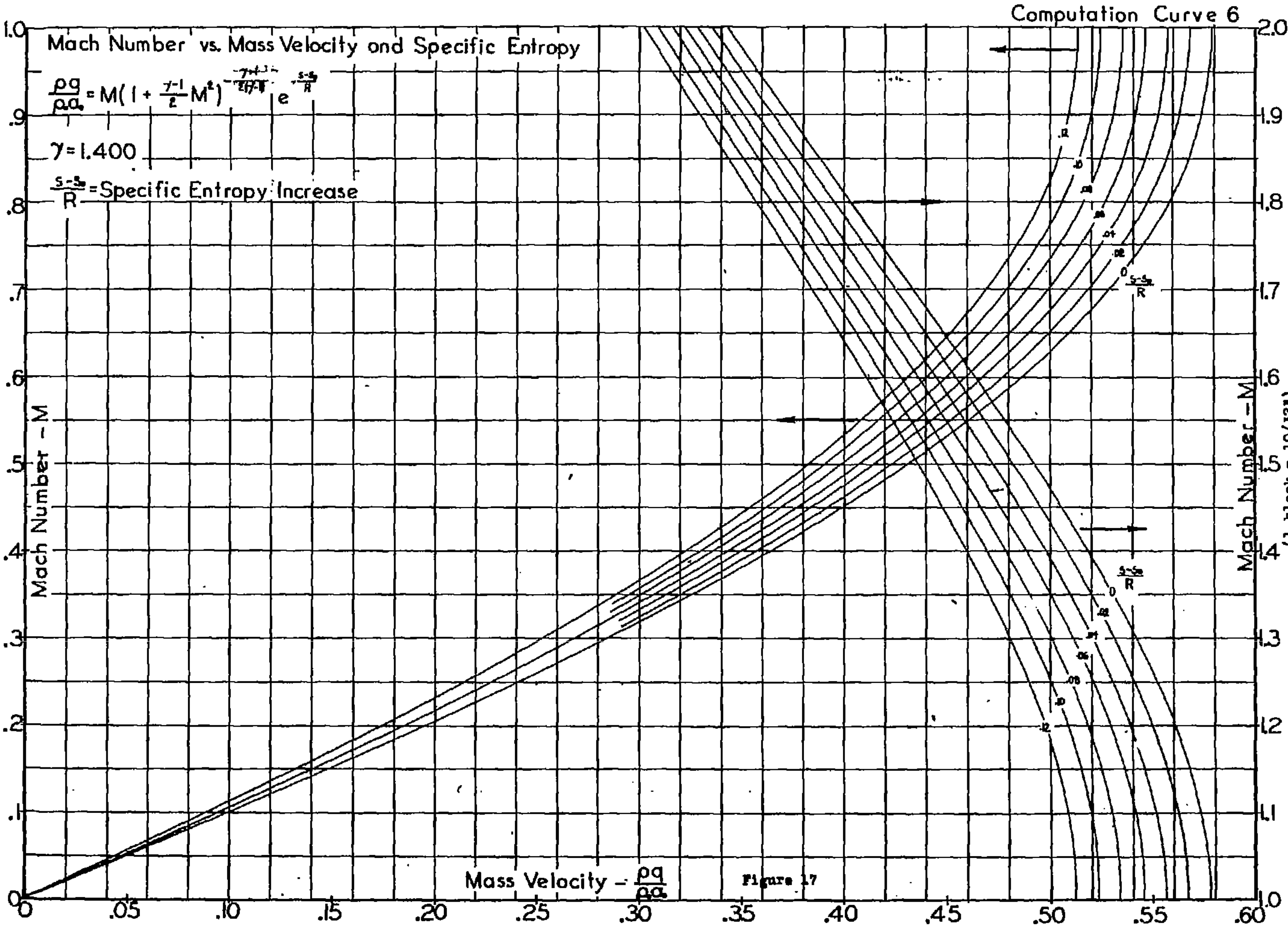
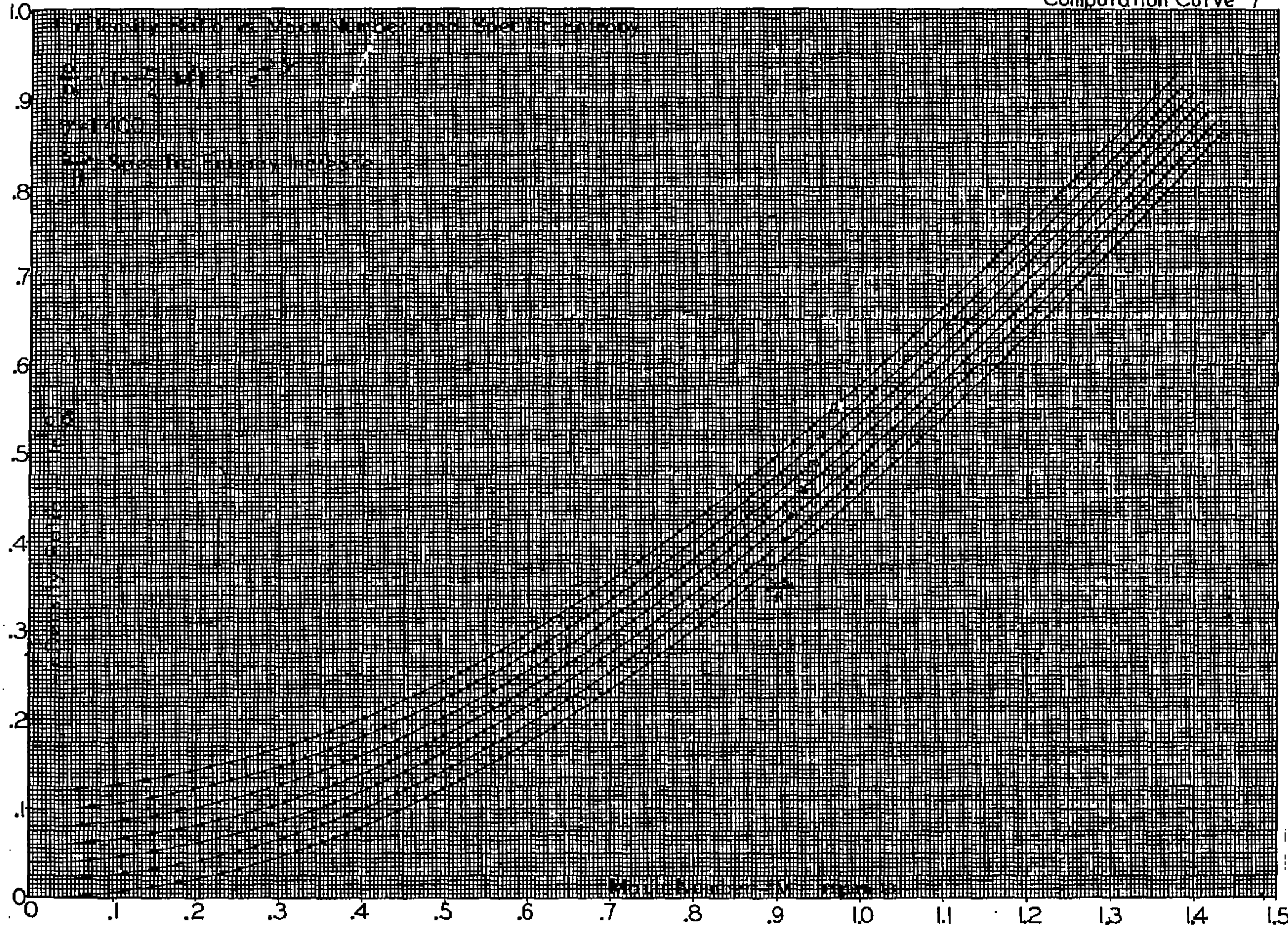


Figure 17

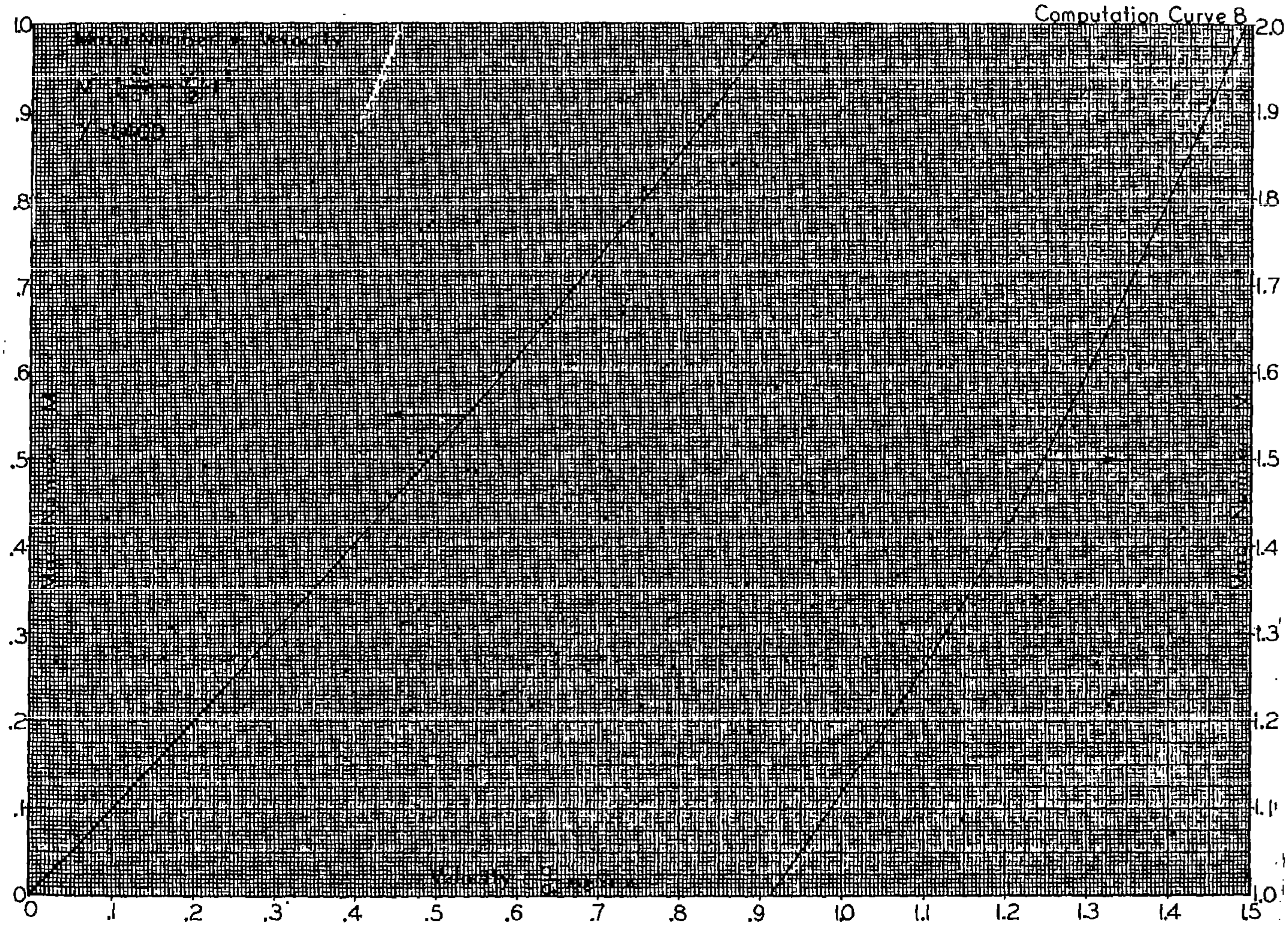
Fig. 17

Computation Curve 7



(1 block = 10/32")

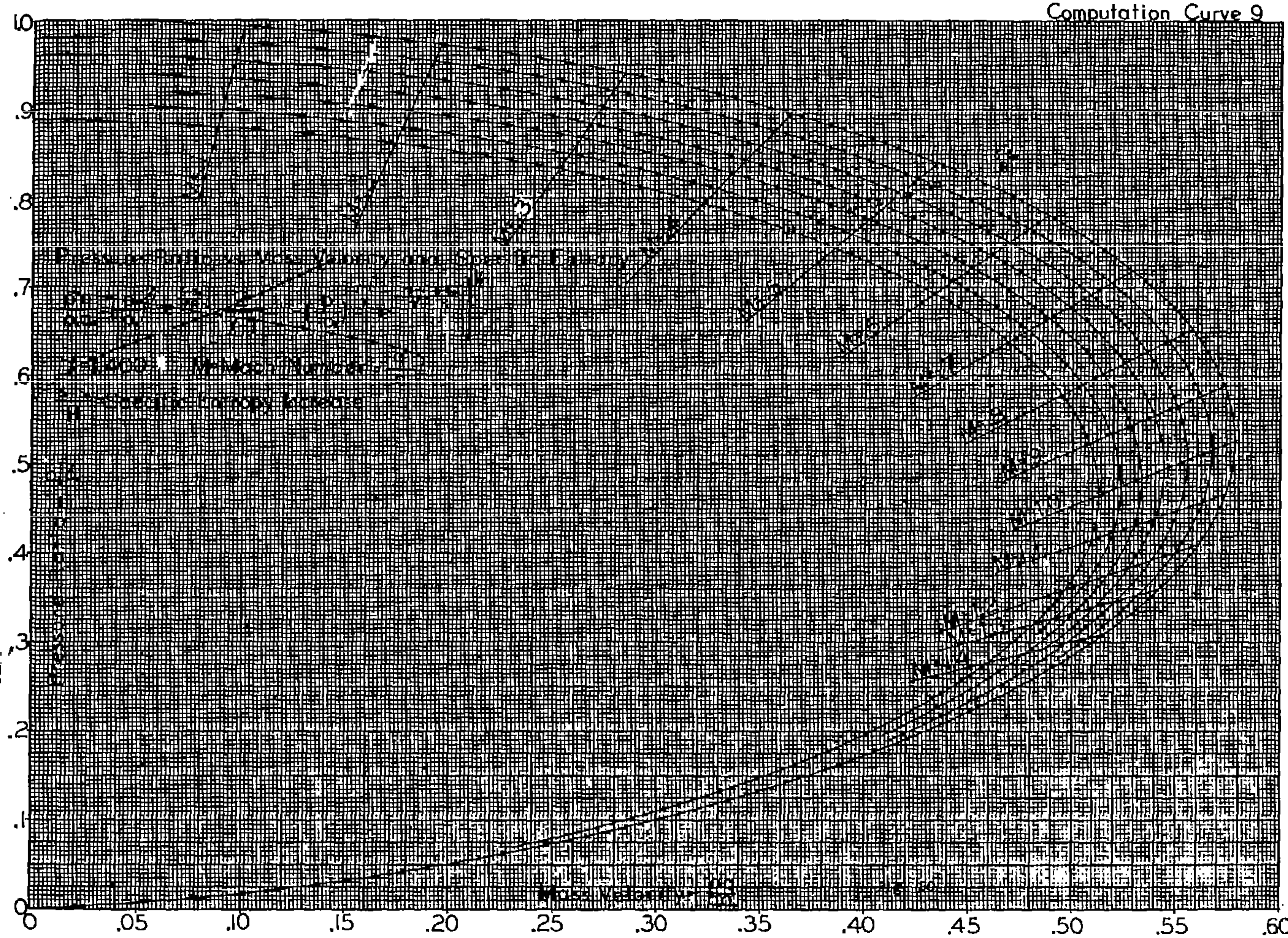
Fig. 18



(1 block = 10/32")

Fig. 19

Computation Curve 9



(1 block = 10/32")

Fig. 20

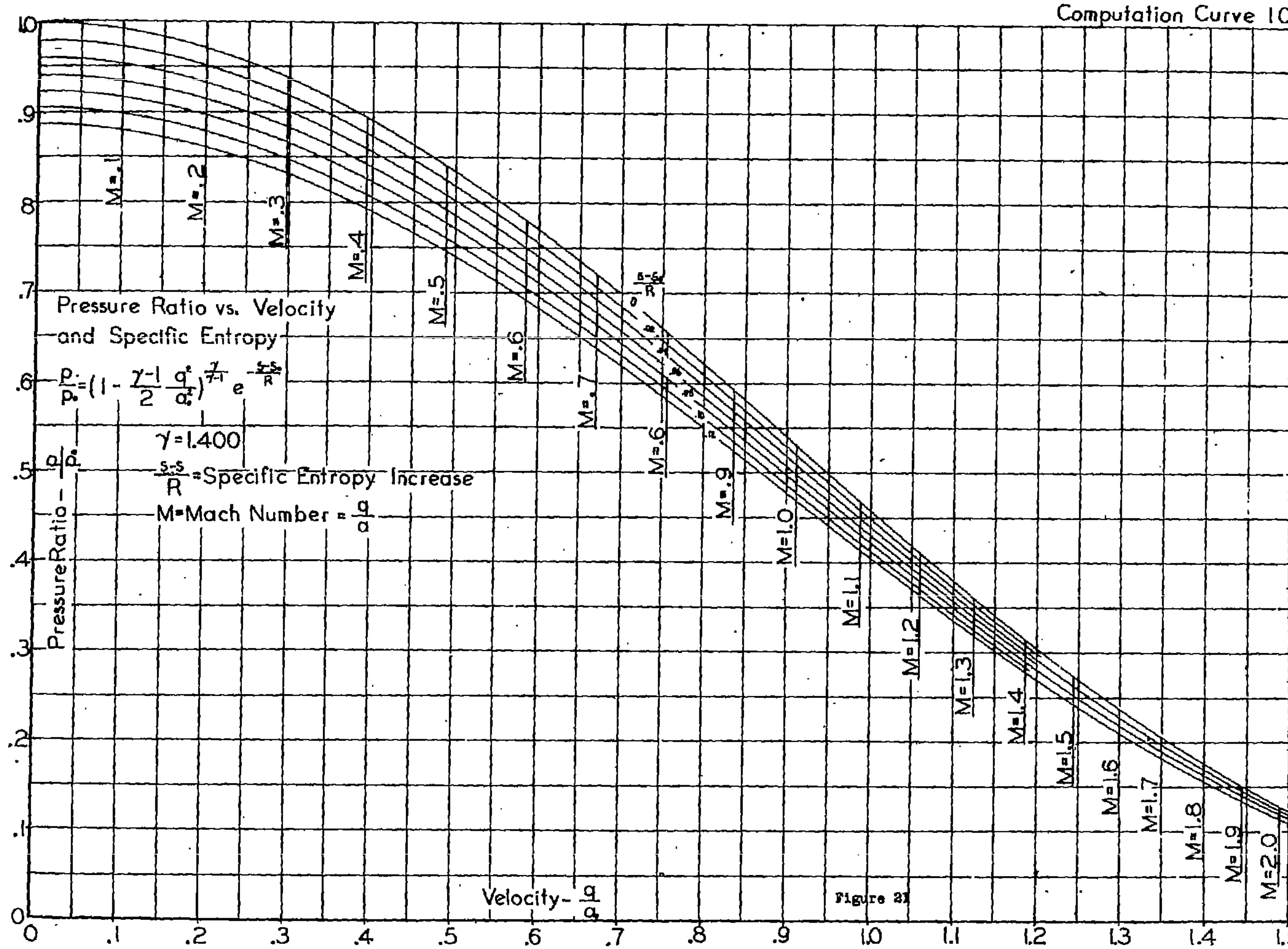


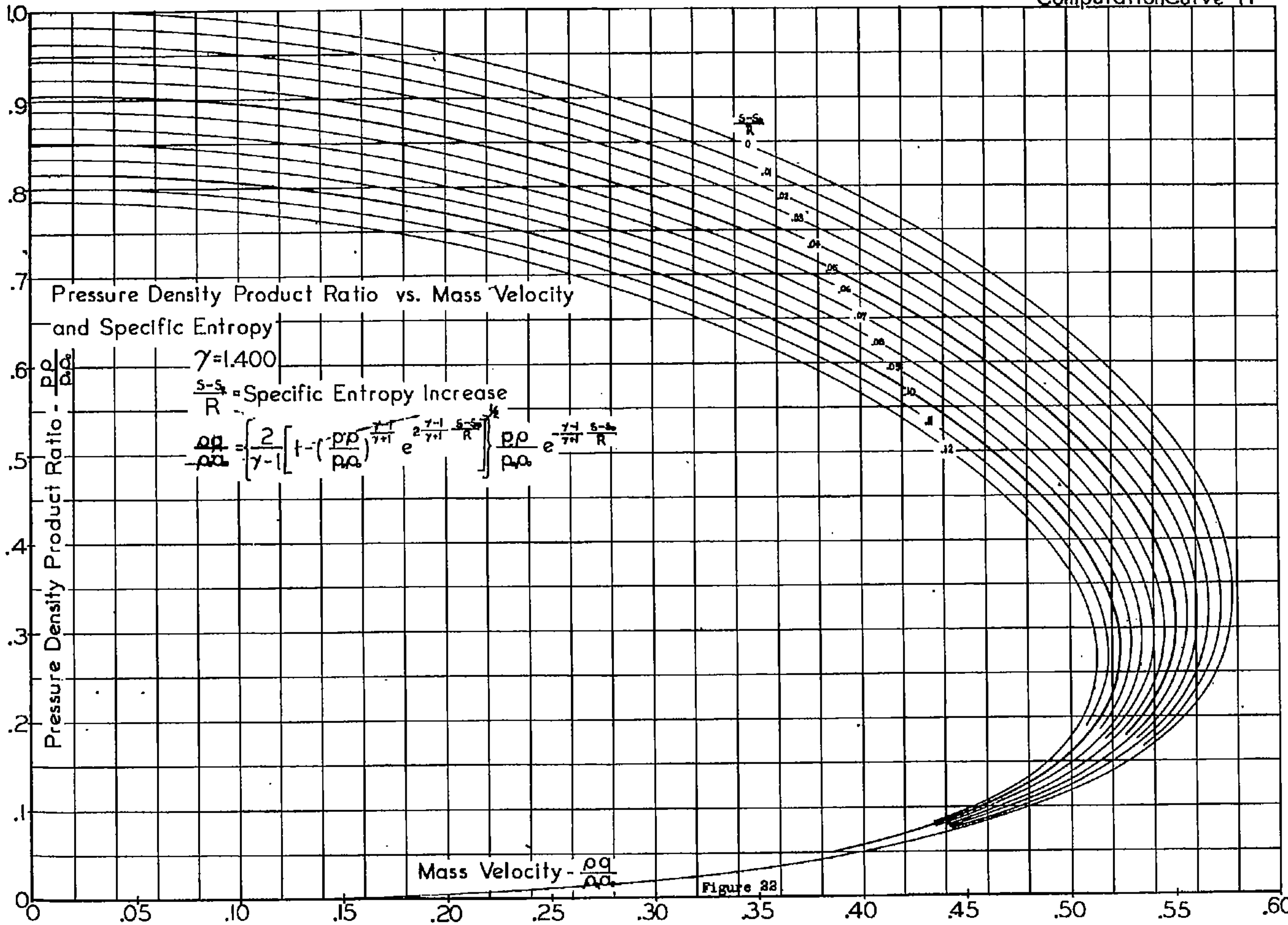
Figure 21

NACA TN No. 838

Fig. 21

(1 block = 10/32")

Computation Curve 11



Computation Curve 12

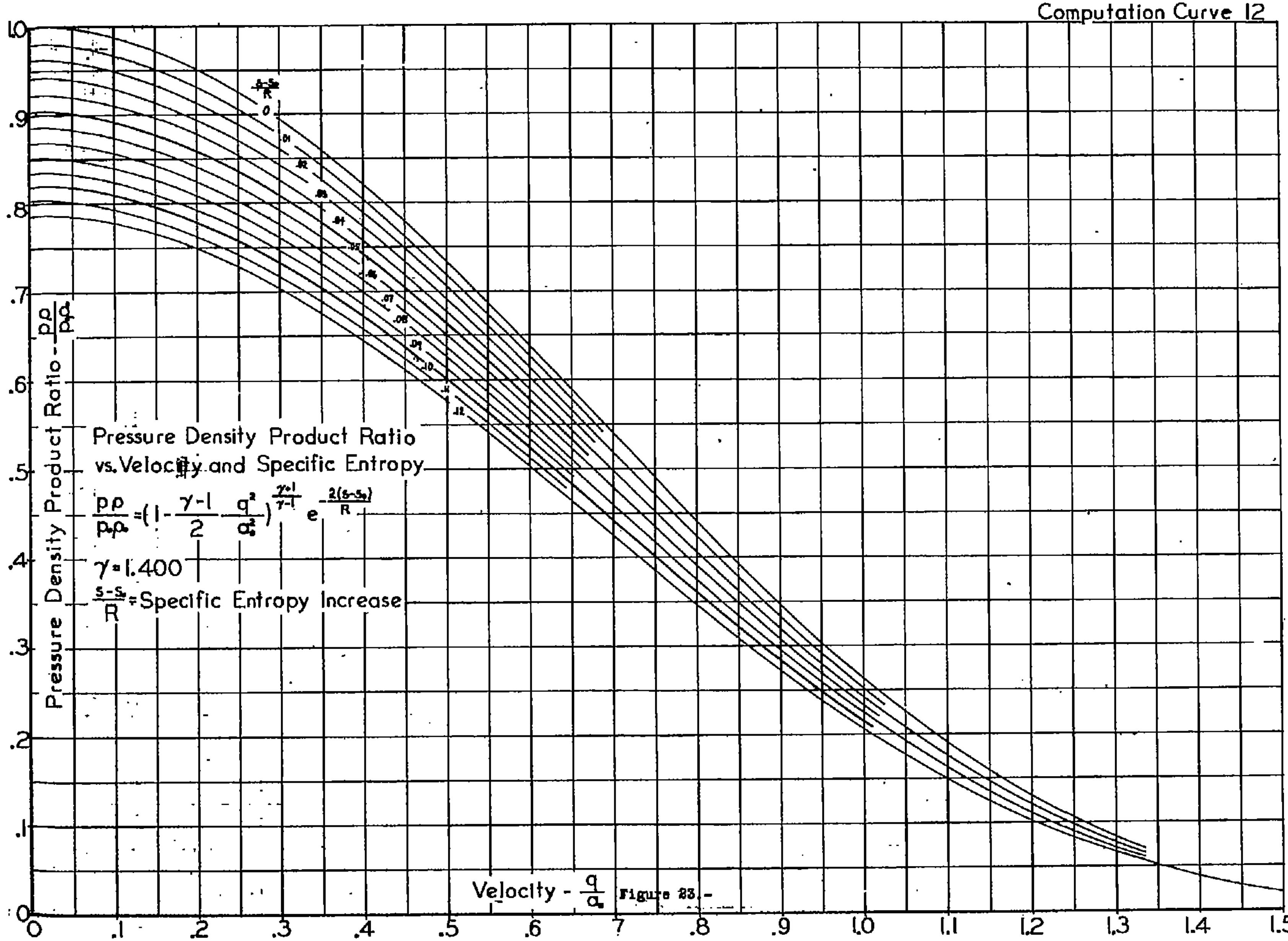


Figure 23.

(1 block = 10/32")

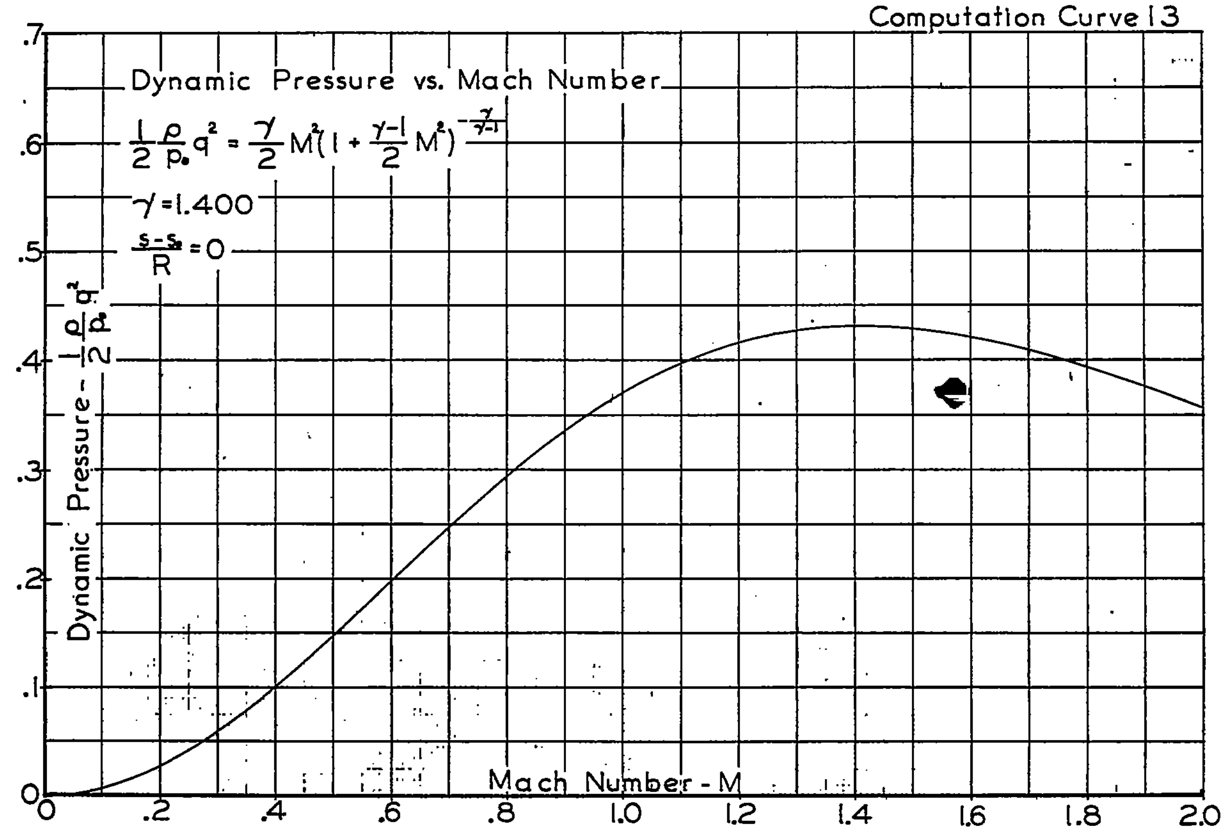


Figure 24.-

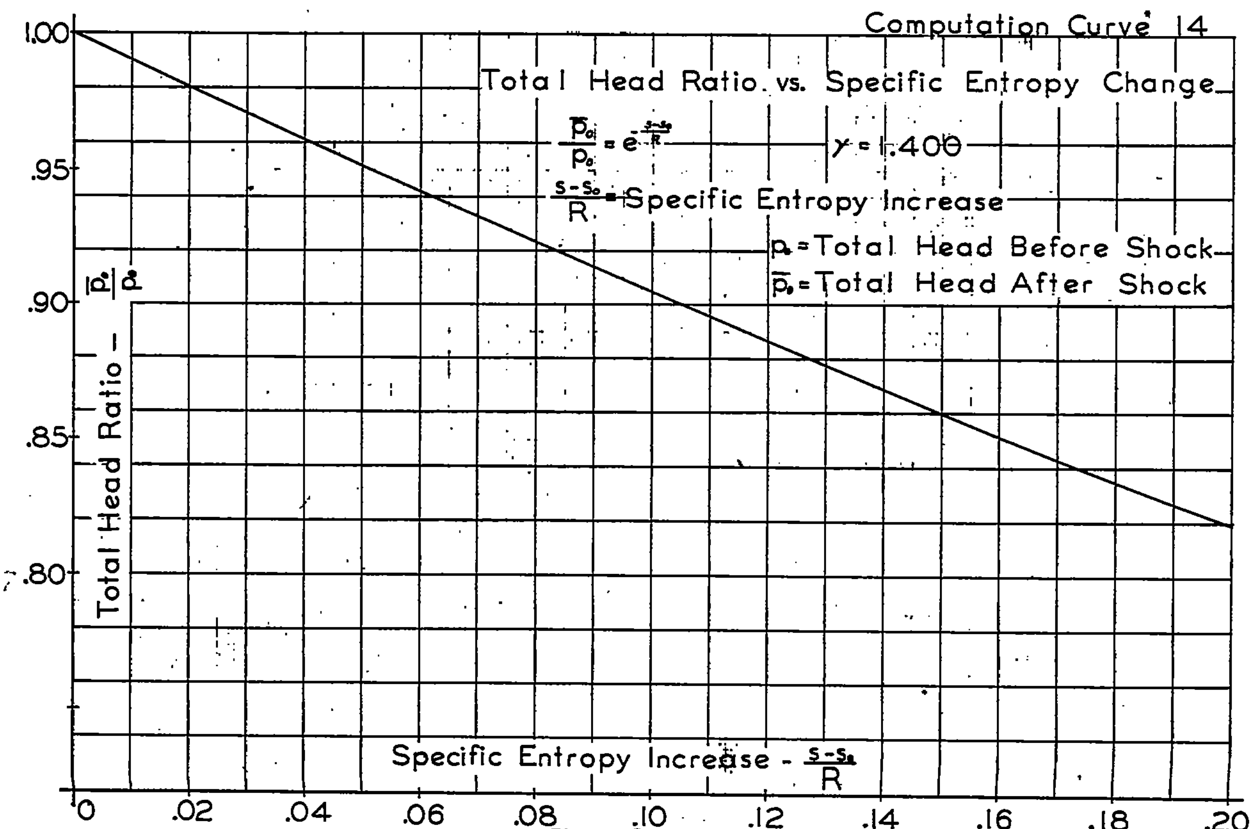


Figure 25.-